Mathematics Direct

THE MATHEMATICAL ASSOCIATION OF AMERICA

The following thirty persons and two institutions have been elected to membership in the Association on applications duly certified:

To Individual Membership

R. G. ARCHIBALD, Ph.D. (Chicago) Asst. Prof., Columbia Univ., New York, N. Y.

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D. M. BROWN, A.B. (Illinois) Asst., Univ. of Illinois, Urbana, Ill.

L. W. COHEN, Ph.D. (Michigan) Asst. Prof., Univ. of Kentucky, Lexington, Ky.

R. H. CONLEE, A.B. (New Mexico) Supt. of Public Schools, Mountainair, N. Mex.

N. C. FISK, D.Sc. (Michigan) Dean, Itasca Jr. Coll., Coleraine, Minn.

F. C. GENTRY, A.M. (Oklahoma) Instr., Tulane Univ., New Orleans, La.

N. C. GRIMES, A.M. (Wisconsin) Prof., Grove City Coll., Grove City, Pa.

MISS BORGHILD GUNSTAD, B.S. (Minnesota) Asst. in Botany, Univ. of Minnesota, Minneapolis, Minn.

E. L. HARP, JR., Asst., Univ. of New Mexico, Albuquerque, N. Mex.

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P. D. JOSE, A.M. (Syracuse) Instr., Syracuse Univ., Syracuse, N. Y.

ROY MACKAY, A.B. (New Mexico) Grad. Student, Univ. of New Mexico; Teacher, High School, Albuquerque, N. Mex.

MAX MARKOWITZ. Undergraduate, Brooklyn Coll., Brooklyn, N. Y.

MISS BUENA C. MATHIS, A.B. (Kentucky) Grad. Asst., Univ. of Kentucky, Lexington, Ky.

L. R. POLAN, A.B. (Milton Coll.) Instr., Alfred Univ., Alfred, N. Y.

W. H. RASCHE, Prof. of Mechanism, Virginia Poly. Inst., Blacksburg, Va.

D. L. ROBB, A.M. (Pittsburgh) Grad. Asst., Univ. of Pittsburgh, Pittsburgh, Pa.

MARGARET M. RYAN, A.M. (Syracuse) Grad. Asst., Syracuse Univ., Syracuse, N. Y.

OLE SCHEY, B.S. (Minnesota) Asst. in Educ. Psych., Univ. of Minnesota, Minneapolis, Minn.

WILLIAM SELL, A.M. (Alabama) Instr., Univ. of Alabama, University, Ala.

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G. W. STREATOR, A.M. (Western Reserve) Instr., Bennett Coll. for Women, Greensboro, N. C.

R. E. STREET. Undergraduate, Rensselaer Poly. Inst., Troy, N. Y.

H. I. TANGJERD, A.B. (St. Olaf Coll.) Instr., St. Olaf Coll., Northfield, Minn.

E. L. WALTERS, B.S. (Washington and Jefferson) Grad. Asst., Syracuse Univ., Syracuse, N. Y.

MARIAN A. WILDER, A.M. (Minnesota) Asst., Univ. of Minnesota, Minneapolis, Minn.

R. T. ZOCH, A.B. (George Washington) Principal Scientific Aid, U. S. Weather Bureau, Washington, D. C.

To Institutional Membership

MENLO SCHOOL AND JUNIOR COLLEGE, Menlo Park, Calif.

OKLAHOMA CITY UNIVERSITY, Oklahoma City, Okla.

W. D. CAIRNS, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The twenty-ninth regular meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the University of Richmond, Richmond, Va., on Saturday, May 9, 1931. Sessions were held in the morning and in the afternoon. Professor Clara L. Bacon, chairman of the Section, presided at both sessions.

One hundred and four persons attended the meeting, including the following forty-one members of the Association: O. S. Adams, R. N. Ashmun, Clara L. Bacon, C. C. Bramble, G. R. Clements, A. Cohen, Alexander Dillingham, J. A. Duerksen, Mary Ewin, P. J. Federico, R. E. Gaines, Michael Goldberg, Patricia Gosnell, Isabel Harris, F. E. Johnston, L. M. Kells, W. D. Lambert, Gillie A. Larew, Florence P. Lewis, B. Z. Linfield, J. J. Luck, G. A. Lyle, Florence M. Mears, Ethel I. Moody, Eugenie M. Morenus, F. D. Murnaghan, E. K. Paxton, R. E. Root, Beulah Russell, T. McN. Simpson, Jr., L. W. Smith, Martha L. Smith, J. M. Stetson, Mildred E. Taylor, John Tyler, C. H. Wheeler, III, G. T. Whyburn, John Williamson, E. W. Woolard, R. C. Yates, Oscar Zariski.

A meeting of the executive committee was held before the opening of the morning session, and during the afternoon session a business meeting was held. The nominating committee appointed at the preceding meeting presented its report, and the following officers were unanimously elected for the coming year: Chairman, E. W. Woolard, George Washington University; Secretary-Treasurer, Paul Capron, U. S. Naval Academy; Members of the Executive Committee, Gillie A. Larew, Randolph-Macon Woman's College, and Oscar Zariski, Johns Hopkins University.

Those attending the meeting were entertained at luncheon by the University of Richmond. During luncheon, the chairman presented Dr. Wilhelm Blaschke, who was a guest at the meeting, and who gave an informal talk. Following the afternoon session, many of those present took advantage of a bus tour of historic points in and around Richmond which had been arranged for them. The Section expressed its great appreciation of the opportunity to meet at the University of Richmond, and of the efforts of Professor Wheeler and his colleagues in arranging for the meeting and providing for the entertainment of the group. It is hoped that this meeting may result in bringing the members from the different regions covered by the Section into closer touch with each other.

Following an opening address of welcome by President F. W. Boatwright, of the University of Richmond, the following seven papers were presented, of which the first five were delivered at the morning session and the last two at the afternoon session:

1. "Primitive roots of prime numbers" by F. E. Johnston, George Washington University.

2. "Rotations in four-dimensional space" by B. Z. Linfield, University of Virginia.

3. "The restricted use of the ruler in constructions of projective geometry" by Oscar Zariski, Johns Hopkins University.

4. "Some applications of Carathéodory's method of geodetic equidistance" by Gillie A. Larew, Randolph-Macon Woman's College.

5. "The potential of a homogeneous cylinder" by Walter D. Lambert, U. S. Coast and Geodetic Survey.

6. "A direct analytical proof of Pascal's theorem and related theorems" by T. L. Wade, University of Virginia, by invitation.

7. "Homologous rectifications of curves" by John Tyler, U. S. Naval Academy.

Abstracts of some of these papers follow:

1. The linear congruence $ax \equiv b \bmod p$, where a and p are relatively prime, has a unique solution. If for a given a, b runs through the $p-1$ totitives of p, there result $p-1$ congruences, whose solutions are the totitives again in some order. This set of equations will define a substitution on the b's, if it is assumed that each constant term in an equation is to be replaced by the solution of that equation. If then a runs through the $p-1$ totitives, there result $p-1$ substitutions which form a regular group, simply isomorphic with the group formed when the totitives are combined by ordinary multiplication. The sum of the numbers in each cycle of each substitution is congruent zero modulo p. If d, a divisor of $p-1$, is of the form p^2k, the group contains a substitution such that each cycle which contains a totitive belonging to the exponent d contains only such totitives. If $d = p_1$, the group contains a substitution involving a cycle consisting of unity and the numbers which belong to the exponent d. If $\sigma(d)$ denotes the sum of the numbers belonging to the exponent d, we thus have $\sigma(p_1) \equiv -1 \bmod p$. By induction we have $\sigma(d) \equiv (-1)^\lambda \bmod p$ if $d = p_1 p_2 \cdots p_\lambda$, where the p_i are distinct primes. These results taken together give a new and simple proof of Stern's generalization of Gauss's theorem which states:

$$\sigma(d) \equiv 0 \bmod p \text{ if } d = p_1^2 K,$$

$$\sigma(d) \equiv (-1)^\lambda \bmod p \text{ if } d = p_1 \cdots p_\lambda.$$

2. In 3-dimensional space it is well known that each 3-dimensional orthogonal matrix whose determinant is 1 determines a rotation leaving a line invariant. In 4-dimensional space, Professor Linfield pointed out, not every 4-dimensional orthogonal matrix whose determinant is 1 represents a rotation, but every such matrix is factorable into a product of two orthogonal matrices each of which represents a rotation leaving a 2-dimensional plane invariant. He also indicated extensions to n-dimensional space.

3. The purpose of this paper is to show that all linear constructions of projective geometry, i.e., all constructions which require only the use of the ruler, can also be performed if the use of the ruler is restricted to the drawing of lines through 4 fixed points A, B, C, D, not all on a line. It is only necessary to show

that given two arbitrary pairs of points, X, Y and X', Y', it is possible to find the point of intersection of the lines XY and $X'Y'$ by means of a construction which involves only lines passing through either one of the 4 fixed points. This is done by considering successively the following cases: (1) The line $X'Y'$ passes through one of the 4 fixed points, say C, the point X is on the line joining two of the remaining 3 points, say on the line AB, and the point Y is arbitrary. (2) The line $X'Y'$ subject to the same condition as in the previous case, while the points X and Y are arbitrary. (3) The points X, Y and X', Y' are taken arbitrarily.

The success of the restricted use of the ruler, as described above, is significant if considered from the point of view of the foundations of projective geometry. It shows that 4 pencils of lines in a projective plane whose centers are not all on a line determine completely *all* the lines of the plane, if we require that the theorem of Desargues and the fundamental theorem of projective geometry should hold. In a sense we can say that when we give all the lines of the plane, we give too much. However it is by no means easy to give a set of assumptions, involving only the 4 given pencils of lines, which should assure the validity of the theorem of Desargues and of the fundamental theorem in the projective plane which is thus determined. Certain necessary assumptions are easily obtained, which are quite similar to those which were introduced by Blaschke in his elegant *Textilmathematik* and by Reidemeister. It can also be seen that if of the 4 centers of the given pencils 3 are on a line, then these assumptions are also sufficient. The general case requires a further investigation.

4. Carathéodory's method of geodetic equidistance is briefly sketched in relation to the simplest problem of the calculus of variations. The method is then applied to the Mayer problem, and the Weierstrass necessary condition is established in a simple and direct fashion. The theory is also used to derive a transversality condition for a certain problem with variable endpoints.

5. In recent experiments at the United States Bureau of Standards for determining the Newtonian constant of gravitation, a cylinder was used as the large attracting mass instead of the usual sphere. This was because it was believed that it would be easier for the machinist to approximate to a mechanically perfect cylinder than to a mechanically perfect sphere. The machinist's difficulties were thus unloaded on the mathematician, who was obliged to derive the necessarily complicated formulas for the attraction of a cylinder on a point close to it. In this paper the formulas used by the Bureau of Standards are derived. These formulas offer certain advantages but in spite of the symmetry about the axis of the cylinder they involve three variables, not all, of course, independent; and in differentiating the potential to obtain the attraction it is necessary to attend to the relations among these variables. Other formulas in terms of the more conventional polar coordinates are also derived. These formulas are an infinite series of terms, each consisting of the product of a coefficient, a power of the radius vector and a zonal harmonic. The argument of the zonal harmonic is the angle between the radius vector and the axis of a cylinder. The coefficients involve

zonal harmonics also, but the argument in these cases is independent of the position of the attracted point and depends only on the proportions of the cylinder.

6. If we interpret the three-dimensional vectors A_1, A_2, A_3 and B_1, B_2, B_3 of the matrices A and B in the equation $B = C \times A$ as the homogeneous coordinates of the vertices of two triangles, the matrix C determines six scalars (the elements in the expansion of $|C|$):

$$I_1 = C_{11}C_{22}C_{33}, \quad J_1 = -C_{13}C_{22}C_{31},$$
$$I_2 = C_{12}C_{13}C_{31}, \quad J_2 = -C_{21}C_{33}C_{12},$$
$$I_3 = C_{13}C_{32}C_{21}, \quad J = -C_{11}C_{23}C_{32},$$

which, as shown by Mr. Aylor in his University of Virginia thesis, are relative invariants under all linear transformations, and the quotient of an I and a J is an absolute invariant. Furthermore, the necessary and sufficient condition that the six points A_1, A_2, A_3 and B_1, B_2, B_3 lie on a conic is that

$$F(I, J) \equiv I_1I_2 + I_2I_3 + I_1I_3 - J_1J_2 - J_2J_3 - J_1J_3 = 0.$$

We will now use the condition that $F(I, J) = 0$, to prove Pascal's theorem.

If the vertices of the hexagon inscribed in a conic are taken in the order $A_1, A_2, A_3, B_1, B_2, B_3$, then the opposite sides are paired as below, the points of intersection being P, Q, R:

$$\left.\begin{matrix} A_1A_2 \\ B_1B_2 \end{matrix}\right\} P \qquad \left.\begin{matrix} A_2A_3 \\ B_2B_3 \end{matrix}\right\} Q \qquad \left.\begin{matrix} A_3B_1 \\ B_3A_1 \end{matrix}\right\} R.$$

Solving the equations of the sides of the hexagon for the points of intersection, we have

$$P = C'_{31}A_2 - C'_{32}A_1,$$
$$Q = C'_{12}A_3 - C'_{13}A_2 \quad (C' = \text{adj. } C)$$
$$R = C_{32}C_{11}A_1 + C_{32}C_{12}A_2 + C_{12}C_{33}A_3.$$

The condition that these three points P, Q, R be collinear, i.e., that Pascal's theorem be true, is that the determinant, Δ, of the vectors P, Q, R shall vanish. By introducing the I's and J's in the expansion of Δ we can write

$$\Delta = |A| \times F(I, J),$$

and consequently Δ is zero, and Pascal's theorem is proved.

Similar proofs of the related theorems of Steiner and Kirkman are likewise based upon the invariant relation that six points lie upon a conic.

EDGAR W. WOOLARD, *Secretary*

FUNCTIONS OF THE MATHEMATICAL ASSOCIATION OF AMERICA[1]

By J. W. YOUNG, Dartmouth College

The American Mathematical Society is devoted to research—it will voluntarily have nothing to do with anything else. This singleness of purpose is doubtless a source of strength to the Society. It would, however, be a source of weakness to mathematics as a whole, and, indeed, to the development of research itself were not other important phases of mathematical enterprise taken care of elsewhere. It makes the definition of the functions of the Association very easy: Everything that is worth doing for mathematics, other than research, is a function of the Association. The latter is the proverbial "George" of the mathematical family.

George is or should be a very busy person. The Association has during its sixteen years of existence done many valuable things. It is not my purpose to pass them in review. Rather would I consider certain new activities which, in my opinion, need to be inaugurated or stimulated.

It may clarify my purpose if I say a word as to the immediate background from which I approach my present task. At the summer meeting a year ago, the Society and the Association appointed a Joint Committee on Funds, which was charged with the investigation of the financial needs of the two organizations and the consideration of ways and means of satisfying them. The Society was and is facing difficult financial problems. The Association has been operating strictly within its assured income; but there are a number of things it wished to do, which it felt it should do, for which additional funds were necessary. The work of this Committee has raised a number of fundamental questions. What is most worth while and most urgent? What is important and why? The financial problems of the Society are largely, though not wholly, connected with the cost of its publications. The research output has been increasing rapidly during the past years; the publications have been correspondingly expanded. If our editorial policy remains the same, these publications will have to continue their expansion. Their cost must then continue to increase. Is this policy sound—or has the time come to raise materially our editorial standards? Are we publishing too much? Should the Society solve its financial problems by retrenchment?

If not, how shall the additional expense be met? What additional sources of support can mathematics find? What are the underlying conditions under which additional sources of support can be tapped? These are a few of the questions which have arisen and they suggest questions still more fundamental. It is some of these more fundamental questions which I wish to raise this afternoon and to invoke the active help of our membership in finding solutions. The questions offer, as you will see, a large variety; their solution will call for varied abilities. Among the more than two thousand members of the Association there surely can be found the men and women who have the ability, the energy, and the enthusiasm to tackle these various tasks successfully. I am issuing a general call

[1] Retiring presidential address delivered at Minneapolis, Sept. 8, 1931.

to service to every member of the Association—let each one find the particular phase of the work for which he or she is best fitted. There is much that needs to be done, and every one can make his or her contribution somewhere.

In the minds of most the Association is thought of as representing the field of collegiate mathematics. Lying between the work of the graduate school on the one hand and the field of secondary mathematics on the other it would inevitably have contacts with both. But circumstances force it actually to penetrate the adjacent domains. Owing to the singleness of purpose of the Society referred to at the beginning of my address, any problem relating to the organization of graduate instruction falls into the proper domain of the Association. I do not propose to raise such problems on this occasion, although they exist. I mention the fact merely to indicate the scope of the work that George may interest himself in. In the field of secondary education, we might expect the National Council of Teachers of Mathematics to carry on. Unfortunately, from some points of view at any rate, the National Council reflects the point of view of our schools of education rather than that of our departments of mathematics. Their official organ, The Mathematics Teacher, seems to me to be almost altogether "teacher" and hardly at all "mathematics." So that here again any real mathematical interests seem to be the function of the Association.

The first question I wish to raise lies perhaps on the border line between the activities of the Society and those of the Association. When is a piece of research important? Every research man has to answer this question, consciously or subconsciously, with respect to his own research. The editors of our journals are consciously facing it all the time. And yet we have no explicitly formulated canons of critical evaluation. We have certain standards of rigor; we insist on originality of result or of method. But beyond this we appear to be guided by little more than an instinctive sense. Literature, art, and the drama develop their critics; why not mathematics? True, we have a kind of criticism in our book reviews; and occasionally indeed a general principle of evaluation may be inferred between the lines of such a review. But this is far removed from the problem of formulating general criteria of evaluation. We are here facing an almost untouched field of inquiry of, it seems to me, great importance.

The problem is an extraordinarily difficult one, which is doubtless one of the reasons why so little has been done with it. Generality is one of the criteria that first comes to mind—generalization has doubtless been one of the most effective modes of progress. May we then infer that the more general a concept, a theorem, a process is, the more important it is? Hardly. Some of you will recall a paper by the late Professor A. D. Pitcher published some years ago in the Bulletin, in which from the point of view of general analysis he sought a generalization of the idea of "continuity." The means employed was the concept of the "development" of the range of the general variable. On the basis of this concept he arrived very naturally at the definition of "continuity with respect to a given development." He then proved the surprising theorem that *every function is continuous with respect to some development of the range.* The result

may be of interest—I am inclined to think it is—but as a generalization of continuity it is obviously futile.

Generalization to be useful is clearly subject to limitations. About *anything* one can say precisely *nothing*, except that the underlying concept is self-contradictory. Unlimited generalization implies not merely futility but paradox. This particular path to virtue leads ultimately to unforgivable sin. Just where along this path should the would-be righteous stop? What precisely, or even vaguely, are the limitations that should be imposed on generality? I do not know.

And generalization is only one of many principles of progress; others raise similar questions. And one consideration shows that any final evaluation of contemporary research is quite impossible. The importance of a piece of research often depends on the unforeseeable future. History is full of examples which show that later developments have resurrected results or processes which for a shorter or longer period had been neglected or forgotten and exhibited them as of prime importance. A notable example will appear presently in a quotation.

Some may feel that our lack of a literature of criticism is a healthy sign. As long as mathematical research is vigorous and progress is rapid, there is no impulse toward critical thought. To such, criticism is a sign of decadence. I do not agree. I think the problem I have suggested is one of great interest and great potential value. Are there not some of our members who will seriously attempt its solution?

In any case the bearing of what I have said on the immediate problems facing the Society should be obvious. Without any more valid canons of evaluation than we now possess, the cutting down on the volume of our publication would be a calamity, which can be justified only on the ground of financial necessity. This, incidentally, is the opinion of the overwhelming majority of those consulted by the Joint Committee on Funds.

The following quotation from a letter received by the Committee from Professor Bell shows how strongly some feel on the proposal, made by a small but responsible group in the Society, to reduce materially the volume of publication by raising the standard of editorial acceptance and relegating the manuscripts which do not meet these higher standards to the waste basket. Professor Bell writes:

"This is the Jehovah complex in its most dangerous form; it is arrogating to oneself prescience and wisdom which some of us still like to think belong only to Almighty God. Who of these men, or of any one else for that matter, now knows what will be looked at fifty years hence, or even ten years hence." Professor Bell after citing a number of other examples refers to the recent work of Ricci. "Until Einstein rubbed their noses in it, the differential geometers ignored Ricci's work and everything that has come out of it, secure in their belief that it was mere formalism because some pundit had so stigmatized it."

In view of some recent discussions I feel that I should add at this point, in order to avoid possible misunderstanding, that what I have said refers solely to

standards of content and not at all to standards of form. It may well be that editors should insist on more careful methods of exposition and that thereby not only could space in our journals be saved but also the value of the papers enhanced.

Related to what I have just been discussing is historical research in mathematics, the fostering of which has long been recognized as a function of the Association. The latter is all set to revive the Bibliotheca Mathematica—it is prevented only by a lack of the necessary funds. May I take this opportunity to voice the hope that our historians and those of us who are interested in historical developments without having as yet qualified as historians, will in the future not confine themselves so exclusively to what might be called the archeology or the ethnology of mathematics. The history of mathematics among the ancients or in the middle ages and the renaissance and among the Chinese and Hindus is doubtless of interest. But I venture the assertion that the history of modern mathematical disciplines is of equal interest and of much greater potential value to present progress.

Here also I may quote to advantage a paragraph from the same letter of Professor Bell: "I am fairly well acquainted," he writes, "with some branches of technical scientific history, and I can state with fair confidence that there is nowhere a work approaching in completeness and usefulness Dickson's *History of the Theory of Numbers*. Americans have made a splendid beginning in this sort of work and, as it was their own idea, they should keep it up . . .. The great usefulness of such histories to research workers is obvious to any one who has used such a history." Here is another important field of work for some of our members to cultivate. Won't some of them enter it?

I come now to the functions of the Association relating principally to collegiate mathematics. The improvement of teaching is largely a question of improving the teachers. The meetings of the Association and of its various sections serve to stimulate enthusiasm and offer opportunity for the exchange of ideas. It would like to expand its missionary work in the less favored sections of the country; it would like to send speakers and lecturers into those districts where stimulation is most needed. This requires additional funds not at present available. The publication of the series of Carus monographs is making brief expository treatment of important topics available. The American Mathematical Monthly through its articles, discussions, and problems provides an additional and effective method of stimulation. The evolution of the Monthly from a small struggling pamphlet to its present size and standing forms an interesting chapter in the development of mathematics in this country. We can all rejoice in its progress and its increased influence. As we look back over this evolution, however, we note a marked shift in its appeal. The Monthly at present is addressed primarily to the college teacher; it used to be of interest to the college student, as well. Nowadays, it has little to offer the undergraduate, except possibly the senior who is majoring in mathematics. Its problem department is in general beyond the ability and training of the undergraduate in his earlier years.

This shift of appeal was doubtless inevitable. The present Monthly is occupying a well defined position and is meeting a well-defined need. I would not be understood as criticizing in any way its editorial policy. I am merely calling attention to the fact that it has, in its progress, left behind a field of usefulness which it used to occupy and which is now being neglected. In other words, I believe there is need for a new periodical devoted to the mathematical interests of undergraduate students.

The embryology of a mathematician is an undeveloped branch of science. Somewhere during his student days some stimulus fertilizes his latent mathematical interests. In rare instances this important event may take place before he enters college; in most cases, however, it occurs during his freshman or sophomore years or later. The nature of this stimulus doubtless shows great variety. It would be interesting and I think not without value, if we could know just what agency kindled the spark and what influences nurtured the flame that started our members on their careers and kept them there, in spite of all rival attractions.

In any case, to attract men and women of ability to our subject is one of our important functions; we do so, presumably, by attempting to arouse and maintain a vital interest in mathematics among those properly qualified. The establishment of undergraduate mathematical clubs is a means to this end. Some of us, perhaps many of us, can testify that the problem department of the Monthly a generation ago was a powerful stimulus in this direction. During my own undergraduate days I was a subscriber to the Monthly and used to have great fun in attempting to solve the problems proposed. The programs of the clubs just referred to show that there are many topics of interest to undergraduates. Occasionally, indeed, papers presented at such meetings would merit publication if a suitable organ were available. Think of the added stimulus in the preparation of such a paper, if the author could look forward to the possibility of its appearing in print.

And so I suggest the publication under the auspices of the Association of a new magazine devoted to the mathematical interests of undergraduates and to others, as will appear presently. A prominent feature of this new magazine would be a department of problems within the range of ability of undergraduates, including some to the solution of which a freshman could aspire. They should be interesting, unusual, not of the conventional text book variety. Such problems exist, though they may not always be easy to find or to invent. The magazine would contain articles of interest to undergraduates, some perhaps, as has been suggested, written by undergraduates. It would take too long to elaborate in detail the plans for such a magazine—nor am I qualified to do so. A further source of articles for such a magazine will appear presently. At this point I wish merely to venture the assertion that this project would probably not require any additional funds. Such a magazine would I believe become self-supporting and might indeed be a source of additional income to the Association. It need not, in fact I think it should not, be a large magazine. Its subscrip-

tion price should be low, not more than $2.00 annually for say ten numbers published during the college year. But its subscribers should be many. We should expect subscribers among the students and faculties of every college of reasonable standing in the country; the mathematical clubs would be our agents. It should get subscribers from among the more enlightened, progressive, ambitious secondary school teachers; it should do for the mathematical interests of such teachers what the Mathematics Teacher does for their pedagogical interests. It would, I venture to say, get subscribers from the general public. The kind of magazine I have in mind would appeal to all those, relatively few in number but in the aggregate numerous, who have mathematical interests. It should be, according to my vision, an agency for the popularizing of mathematics. But of that more presently—I am getting a bit ahead of my story.

Does the project strike a responsive chord in the minds and hearts of some of you? Among our more than two thousand members does the man or woman exist who has the ideals, the enthusiasm, the courage, the tact, the organizing ability to carry this project to success as editor-in-chief? Do we have in our membership the men and women able and eager to cooperate? It is a large and difficult project; it will have to command the enthusiastic and active support of a considerable number to make it go.

I referred a moment ago to the popularizing of mathematics. I fancy that some will be repelled by the suggestion. They will feel that it can't be done; or at least that it can't be done without loss of dignity, without loss of scientific accuracy. And mathematics, they will say, without precision is not mathematics. I will grant the latter. I will also grant that popularization must be used in a relative sense; some degree of familiarity with and interest in elementary mathematics must be presupposed on the part of the prospective reader. But this much being granted the production of a relatively popular mathematical literature is possible and highly desirable. That it is possible in such abstract domains as modern mathematical physics is shown by the recent books of Whitehead and Eddington; in the domain of pure mathematics, by that of Professor Danzig. That there is a considerable demand for such books on the part of the general public is shown by the sale which these books have enjoyed.

Some of you are perhaps familiar with the recent delightful little book by Toeplitz and Rademacher entitled "Von Zahlen und Figuren"—in which a large variety of topics from the domain of advanced mathematics are discussed with accuracy and clarity, on the basis of a minimum amount of technical mathematical training. You are all familiar with Klein's "Elementarmathematik vom höheren Standpunkte aus." I am here suggesting the opposite point of view as offering a valuable and important field of enquiry: "*Advanced mathematics from the elementary point of view.*" We have done little with this field in this country —and it is worthy of our best efforts.

It is, indeed, a field not without real scientific interest. May I illustrate what I mean by a personal experience. At a recent meeting in New York the program committee organized a symposium on "Recent work in Differential Geometry;"

one of the lecturers on that occasion was Professor Struik and his topic was "Differential Geometry in the Large"—a topic that has been developed since about 1912. It is, as most of you know, concerned with differential properties of curves and surfaces as a whole rather than at a point—it is in fact in its present stage largely concerned with properties of ovals and their counterpart in space. The theorem that every oval has at least four vertices—i.e. points at which the curvature is either a maximum or a minimum—is typical. Many interesting theorems relate to ovals of constant width. Professor Blaschke, who spent some months in this country last year, is one of the principal proponents of this domain.

I was greatly interested in Professor Struik's lecture at the symposium. He gave a very elegant proof of the four-vertex-theorem using the idea of a vector integral. I had an instinctive feeling that this and other theorems on ovals should, on account of their simplicity, be capable of more elementary treatment. A few weeks later I had succeeded in proving the four-vertex-theorem and a considerable number of theorems about ovals of constant width as limiting cases of equiangular polygons. Most of the latter theorems were new to me and they seemed distinctly amusing. An examination of the literature showed that they had all been derived by more advanced methods between the years 1912–1916. My methods were almost ridiculously elementary—I used nothing more advanced than elementary trigonometry to prove my theorems on polygons—a college freshman could follow the argument—and as to the limiting processes I needed only the fact that the limit of $\sin \theta/\theta$ is equal to unity. It was originally my intention to make these developments the subject of my present address—I think you would have found them entertaining.

What I have just described suggests another remark concerning the subject of critical evaluation which I discussed earlier. I wonder if we do not often measure the importance of a result by the difficulty of its derivation. Suppose, what is quite possible, that some obscure teacher in a college or high school had by the methods I used derived the theorems just referred to, say twenty years ago, when they were new. Would our editors have considered them important? Would the Transactions, for example, have accepted the paper? I seriously doubt it. Would they not have said "Very pretty—but hardly up to our standard. Send it to the Monthly," or words to that effect. Perhaps they would have been right—perhaps these results are merely pretty and not really important. How are we to tell? In any case, however, they are important or not in themselves and not on account of the method of their derivation.

To return to my original topic. My illustration is a good example of what I mean by "advanced mathematics from an elementary point of view." Does anyone believe for a moment that it is an isolated example? There must be many others awaiting discovery. Some years ago Professor Coolidge gave an address to our undergraduate club at Dartmouth in which he derived many of the geometric properties of the cycloid very easily by elementary geometric methods without the use of the calculus.

Here then is another large field of inquiry which I wish to call to the attention of our members as very worthy of cultivation. As I have indicated it seems to be of real scientific value—especially to those of us who have a perhaps irrational faith that mathematics is essentially simple—that complexities are temporary blemishes which future insight will remove. Its bearing on the popularization, relatively, of mathematics is obvious.

There is another aspect of popularization that is of tremendous importance from the practical point of view of securing financial support for mathematical enterprise. I refer to the popularization or perhaps, better, publicising of the value of mathematical research. The greatest obstacle that lies in the way of securing funds is popular ignorance on this subject. The educated layman knows that mathematics is an important subject, that it has important applications in engineering, in applied science, and presumably in other fields. A small proportion are perhaps sufficiently well informed to realize, if they happen to give the matter any thought, that our civilization is largely built on a mathematical foundation; that, in its material manifestations at least, it could not have arisen and could not continue to exist without mathematics. But there are very few who have any conception of mathematics as alive and growing. To practically every "man-on-the-street"—including of course Wall Street—the practical aspects of mathematics were developed long ago. He has no conception of the fact that the research of today becomes the application of tomorrow, if indeed he has any realization that there is any research today.

This ought not to be. Whatever may be the opinion as to the feasibility of popularizing modern mathematics itself, there can be little doubt, it seems to me, that the tremendous importance of mathematics can be demonstrated even to the man-on-the-street if sufficient effort is made by those qualified to make the demonstration. The influence of mathematics on the various aspects of our evolving civilization, cultural as well as technological, should make a fascinating story if told by men of vision and imagination with the necessary literary gifts. May we not hope that such individuals will emerge from among our members, if the need is once realized. There is unquestionably a market for works dealing authoritatively and interestingly with the various aspects of this subject. The author of such a work could look forward to financial rewards as well as to the satisfaction of contributing materially to the advancement of mathematics. Articles discussing limited portions of this vast subject would be appropriate for inclusion in the proposed new periodical.

Would you consider me quite mad, if I suggest that this new periodical might find a sale among the general public; that we might find it ultimately on the news stands side by side with The Scientific Monthly and The Scientific American? I wonder. There is evidence to indicate that there is a far greater popular interest in our subject than we are generally inclined to believe.

Articles on the influence of mathematics could also find a place in magazines of general appeal and even in our Sunday newspapers. Numerous specific problems that engage the interest of mathematicians are also capable of being pre-

sented in such a way as to arouse popular interest. Dr. Fry has recently expressed his conviction that the problem of the man, the tree and the automobile, which was the subject of a recent paper in the Monthly by my colleague C. E. Wilder, could have commanded a full half-page in the Sunday edition of our metropolitan newspapers, if properly written up for this purpose.

Mathematics needs more publicity. The American Mathematical Society and the Mathematical Association of America should have more publicity. Dr. Fry in a recent letter called attention to the fact that, while Professor Einstein recently spent several months in this country, neither the Society nor the Association seems to have taken any official notice of the fact.

I believe the officers of the Association are at Dr. Fry's suggestion considering the possibility of remedying this situation by the appointment of a permanent committee on publicity.

I come to my final suggestion. Some of you doubtless heard the address delivered at the last Christmas meeting of the Association in Cleveland by Professor Radó on "Mathematics in Hungary." He there told of a mathematical competition held annually among selected graduates of Hungarian secondary schools. It is a national event which is given much publicity and attracts much attention. Moreover it has real scientific value—about half of the winners of this competition, I am told, have become mathematicians of note—a remarkable proportion. Could we not, in this country, emulate the example of Hungary?

The problems proposed for such a competition must be such that they are solvable on the basis of the information and training acquired by a high school graduate—i.e., they should require for their solution, in the way of technical training, nothing beyond plane trigonometry and advanced algebra—but they should be such as to call for unusual originality, power and insight.

I know nothing as to the machinery employed in Hungary for the purpose of this competition. Hungary is a relatively small country; the problem there would be simpler than with us. But I wonder if the already existing machinery of the College Entrance Examination Board could not be utilized for such a purpose? As to the cost of such a nation-wide competition, would not each secondary school from which candidates come be willing to pay the necessary fee? This is not the place to consider the project in detail—it seems to me well worth the most careful consideration as to its feasibility. It would give mathematics and the organization which sponsored it wide and favorable publicity. Would it not be worth while to appoint a committee to study its possibilities?

Appreciation on the part of the general educated public of the importance of mathematical activity would not only make easier the securing of material support for our enterprise—it would unquestionably help in recruiting our ranks with promising material. I have often been struck with the number of students of unusual ability and promise who are drawn away from a mathematical career by the rival attractions of business and the other professions. This is due, in part at least, to the present attitude of the public toward mathematics. The right sort of publicity would materially improve this situation.

I am at the end of my address. You have doubtless observed that my title, The Functions of the Association, is to some extent a misnomer. While I have suggested some projects that the Association as an organization might sponsor, a good deal of what I have said is directed to individuals and calls for individual effort. These portions of my remarks might have been entitled "Unsolved problems of mathematics—other than research."

We can't all be research men. Some of us do not even want to be. I have sought to show that there exist wide fields of enquiry and activity other than research that are important, interesting and worthy of intensive cultivation, and that are being sadly neglected. I have incidentally attempted to combat the attitude, if and wherever it exists, that would make of research a fetish, that proclaims that the only worthy function of a mathematician is research and that other activities are to be looked on with contempt. There is fortunately very little of this sort of self-righteous snobbery in our two organizations. The great majority of our research men themselves realize that the roots and trunk and branches of the tree of mathematics are quite as important as the blossom or the fruit; and that the former must exhibit healthy life if the latter are to be produced at all. Any other attitude is so utterly stupid as hardly to merit attention. To change the simile the star back on a football team who advances the ball and receives the plaudits of the multitude knows, unless his head has been completely turned by an exaggerated sense of his own importance, that he would be quite helpless were it not for a strong line and able interference.

It is probably impossible to determine the relative value of a guard and a back on a football team. It is probably even more difficult to determine the relative value, to the mathmatical organism as a whole, of the research man and the man who labors to improve the conditions which make research possible and which give it significance. Both are essential. But I am quite clear in my own mind, that if entry into the mathematical heaven depends on what a man has done for mathematics during his life on earth, the record of such a man as our good friend Slaught will far outweigh that of most mere research men. There is important work for all of us. The sin of the mathematician is not that he doesn't do research, the sin is idleness, when there is work to be done. If there be sinners in my audience I would urge them to sin no more. If your interest is in research, do that; if you are of a philosophical temperament, cultivate the gardens of criticism, evaluation, and interpretation; if your interest is historical, do your plowing in the field of history; if you have the insight to see simplicity in apparent complexity, cultivate the field of advanced mathematics from the elementary point of view; if you have the gift of popular exposition, develop your abilities in that direction; if you have executive and organizing ability, place that ability at the disposal of your organization. Whatever your abilities there is work for you to do—for the greater glory of mathematics. And this, I think, is the nearest I have ever come to preaching a sermon.

A NOTE ON THE CIRCULAR CUBIC REGARDED AS THE ENVELOPE OF FAMILIES OF CIRCLES

By H. G. GREEN, and L. E. PRIOR, University College, Nottingham, England

The investigation of the circular cubic and its enveloping circles usually consists of a direct attack which involves heavy and rather artificial analytical processes,[1] or else takes the form of a deduction from the properties of a quartic curve.[2] A third powerful method is that of a projection from three dimensions,[3] but all these methods have the slight disadvantage that they may lead some to believe that the problem is incapable of simple direct treatment. We give a discussion based on the methods of pure geometry and involving only the elementary properties of the cubic curve. It also shows clearly the development of the circles.

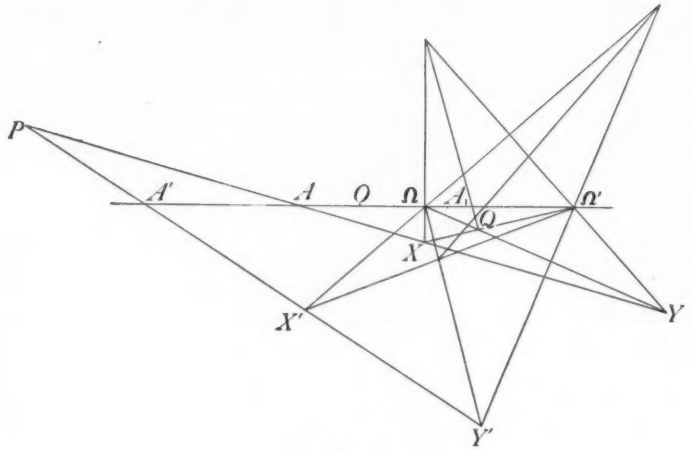

FIG. 1

We consider only the general case of the circular cubic. The modifications for special cases can be followed through on the same lines. Taking Ω, Ω' as the circular points let the circular cubic cut the line at infinity again at O; let P be the point of contact of one of the four tangents from O to the curve. Through P draw two straight lines cutting the curve again in X, Y; X', Y', respectively. Then since the three sets of points on the cubic Ω, Ω', O; X, Y, P; X', Y', P are collinear, and OP is a tangent, the points X, Y, X', Y', Ω, Ω' lie on a conic, or in other words the points X, Y, X', Y' are concyclic. If we keep the line PXY fixed and allow $PX'Y'$ to approach it indefinitely, the circle becomes in the limit the circle which has double contact with the cubic at X and Y.

[1] Hilton, *Plane Algebraic Curves*, p. 217 et seq.

[2] Basset, *Cubic and Quartic Curves*, pp. 152–159.
Salmon, *Courbes Planes*, p. 350.

[3] Baker, *Principles of Geometry*, Vol. 4, p. 97.

Taking any position of $PX'Y'$, the power of P for the circle of contact on this chord is equal to that for the circle of contact on XY, since X, Y, X', Y' are concyclic. Hence for all positions of $X'Y'$, the circles of contact are orthogonal to a circle with centre P, and radius $\sqrt{(PX \cdot PY)}$. The cubic is therefore the double envelope of a family of circles cutting the fixed circle with centre P orthogonally, and further both the cubic and enveloping circles are self-inverse with respect to the fixed circle.

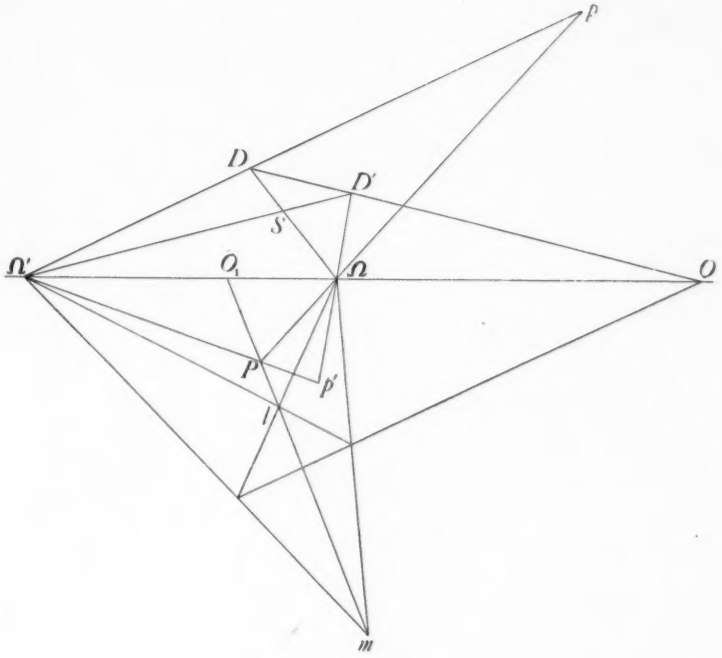

FIG. 2

In the non-singular case four families of enveloping circles can be built up in this way. If P_1 and P_2 are two positions of P, r_1 and r_2 being the radii of their circles of inversion, and if P_1P_2 meets the cubic again in Z, then from the inversion properties

$$r_1^2 = P_1Z \cdot P_1P_2, \quad r_2^2 = P_2Z \cdot P_1P_2,$$

and therefore

$$r_1^2 + r_2^2 = P_1P_2^2,$$

or the circles of inversion are mutually orthogonal.

Consider two near positions PXY, $PX'Y'$ cutting $\Omega\Omega'$ in A and A'. The polar of A with respect to the circle $\Omega\Omega'XYX'Y'$ is the harmonic line of the

quadrangle $\Omega\Omega' XY$ not passing through A. Hence the centre of the circle (the pole of AA') is the intersection Q of the two harmonic lines of this type (see Figure 1). In the limiting case, Q is the point of contact of the harmonic line with its envelope, and is also the centre of the circle touching the cubic at X and Y. Consider now the tangents to the locus (not coinciding with $\Omega\Omega'$) at any point A_1 on $\Omega\Omega'$. The corresponding position of PXY must pass through its conjugate A with respect to Ω, Ω', and is therefore fixed. Hence the tangent from A_1 to the locus is unique, excepting the possibility of the tangency of $\Omega\Omega'$. Take positions of PXY approximating to PO. As it approaches PO the tangent to the envelope considered above approaches $\Omega\Omega'$, and the point of contact approaches O_1, the conjugate of O with respect to Ω, Ω'. Hence the locus of Q is a parabola, whose axis passes through O_1 (see Figure 2). If we take $P\Omega$ as the line PXY, the harmonic line of the quadrangle becomes ΩS, where S is the singular focus of the cubic. Similarly $\Omega' S$ is a tangent to the Q locus, or S is the focus of the parabola. Further if $P\Omega$ cut the cubic again in p and $\Omega' p$ meets $S\Omega$ in D, then D is the point of contact of the parabola with $S\Omega$. Similarly D' may be constructed on $S\Omega'$, and the directrix is DD', passing through O since it is at right angles to SO_1. For the tangent at the vertex let PO_1 meet the cubic in l and m. Then the tangent at the vertex is the harmonic line, passing through O, of the quadrangle l, m, Ω, Ω'.

If two tangents to the cubic from Ω and Ω' be so chosen that the join of the points of contact passes through P, (which is possible in four ways) and if these tangents intersect in s, then Ωs, $\Omega' s$, or equally well the point circle s, can be considered as a circle of the enveloping system. Since the enveloping circles cut the circle of inversion of the same system orthogonally it follows that the ordinary foci s lie four by four on the four circles of inversion.

THE LEMOINE CENTER IN THE GEOMETRY OF THE TETRAHEDRON

By ALBERT A. BENNETT, Brown University

§1. In the geometry of the triangle among the points appropriately called centers of the triangle are:

1. The centroid, which is the point of intersection of medians of the base triangle.

2. The circumcenter, which is the center of the circle circumscribed about the triangle.

3. The incenter, which is the center of the inscribed circle.

4. The orthocenter, which is the point of intersection of the altitudes.

5. The symmedian or Lemoine center, which is, simultaneously, (i) the isogonal conjugate of the centroid, (ii) the center of perspectivity of the given triangle and the triangle whose sides are tangent to the circumcircle at the vertices of the base triangle.

While the definitions in 2, 3, 4, 5(i) are essentially metrical, it is significant that the definitions 1, and 5(ii) remain covariant under more extensive groups. The centroid is determinate under the general affine or "parallel" group of transformation upon the base triangle, and the Lemoine center defined in 5(ii) is covariant under the most general projective group which leaves the base triangle and its circumcircle unaltered.

For a study of the triangle and its circumcircle in relation to purely projective properties, it is convenient to make use of known Euclidean results so as to render intuitively obvious, relations otherwise unfamiliar. The Lemoine center is the center of a perspectivity whose associated axis of perspectivity is called the Lemoine axis. If Euclidean language be used appropriate to treating this line as the line at infinity, then the triangle may be interpreted as a regular triangle, with Lemoine center as its center of symmetry, while the circumscribed circle, continues to be treated as a circle in this metric. The topic is almost devoid of interest due to its extreme simplicity.

The study of the properties of three distinct points under the inversive group leads to the isodynamic point-pair, the Apollonian circles and other elements of interest, but does not provide a unique "center," and for this reason neither this question nor its space analogues will be discussed here.

§2. In studying the geometry of the tetrahedron,[1] it is natural to look for extensions to three-space of relations which have proved significant in the plane. Taking up in turn the five centers mentioned above, it is easily verified that for the tetrahedron one has:

1. The centroid. This is the point of intersection of the four medians, each drawn from a vertex of the tetrahedron to the centroid of the opposite face and

[1] For Desmic tetrahedra, see

R. W. H. T. Hudson, *Kummer's Quartic Surface*, Cambridge, University Press, 1905. Chapter 1, and references.

For mensurational relations of the metrical tetrahedron and some other relations also, see

Hermann Thieme, *Die Elemente der Geometrie*, Leipzig, Teubner, 1909, pages 360–375.

Richard Baltzer, *Déterminants* (Translation by Hoüel), Paris, 1861, pages 189–211.

G. Doster, *Déterminants*, Paris, 1883, pages 252–293.

Gustav Holzmüller, *Elemente der Stereometrie*, vol. 2, Leipzig, 1900, pages 186–251.

For a detailed discussion of the "parallel geometry" in space, see

Lothar Heffter, *Lehrbuch der analytischen Geometrie*, vol. 2, Leipzig, Teubner, 1923, pages 196–270.

For the classical geometry of the tetrahedron, see

J. J. Neuberg, *Zur Tetraedergeometrie*, Leipzig, 1909, (Reprinted from Archiv der Mathematik und Physik (3) vol. 16 (1909).

Eugène Rouché et Ch. de Comberousse, *Traité de Géométrie* (8th edition), Paris, 1912, vol. 2, note 4. *Sur la Géométrie récente du tétraèdre*, pages 643–564.

J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, pages 233–240.

George Salmon, *Geometry of Three Dimensions*, 5th edition, 2 vols. (1912 and 1915), London, Longmans, Vol. 1, pages 138–145.

E. Duporq, *Premiers principes de géométrie moderne*, Paris, 1912, pages 101–110.

Cl. Servais, *Sur la géométrie du tétraèdre* (5 articles), Acad. Roy. de. Belg. (Bull Cl. Sci.) (5) vol. 15 (1929), and vol. 16 (1930).

is situated one-fourth of the distance from the face. It is also the point of intersection of six planes, one through each edge of the tetrahedron and through the mid-point of the opposite edge. It is also the common mid-point of each of the three lines joining mid-points of opposite edges. It is also the mid-point of each of the three parallelograms with vertices at the mid-points of two pairs of opposite edges.

2. The circumcenter, which is the center of the sphere circumscribed about the base tetrahedron.

3. The incenter, which is the center of the sphere, inscribed in the base tetrahedron.

4. The orthomedial center. The four altitudes of the base tetrahedron are not in general concurrent. They are four rulers of an equilateral hyperboloid of one sheet, whose center is the orthomedial center or Monge Point (Monge 1808). It is also the point of intersection common to the six planes, each through the mid-point of an edge, and perpendicular to the opposite edge. Save for the specialized orthocentric tetrahedra for which the hyperboloid reduces to a cone, there is no "orthocenter," although the orthomedial center is an appropriate generalization to space.

Even the fact that for the plane case, the centroid, circumcenter and orthocenter are collinear (on the Euler line) with simple numerical relations for their mutual distances has a space analogue since the distance of the orthomedial center of the tetrahedron, to the circumcenter is bisected by the centroid.

5. The symmedian center, which is the isogonal conjugate of the centroid. However the symmedian center so defined does not enjoy projective properties analogous to those holding in the plane. The tetrahedron circumscribed about the circumsphere and tangent at the vertices of the base tetrahedron, will ordinarily fail to be centrally perspective with the base tetrahedron. The four corresponding joins are rulers of an hyperboloid.

The existence of a projectively determined point, which we shall call the Lemoine center of a tetrahedron, which enjoys properties generalizing in a sense the definition 5(ii) in the plane, has not been hitherto noted. The symmedian center, certainly fails to fulfill these demands. Thus we shall for tetrahedra distinguish between the symmedian and the Lemoine centers, although for triangles these coincide.

§3. Before discussing the Lemoine center, it is well to call to mind some elementary projective properties of the tetrahedron with reference to a general point P not on one of its face-planes. Let the vertices of the base tetrahedron be called the base points, in tetrahedral coordinates we have $B_1=(1, 0, 0, 0)$, $B_2=(0, 1, 0, 0)$, $B_3=(0, 0, 1, 0)$, $B_4=(0, 0, 0, 1)$. Let $P=(p_1, p_2, p_3, p_4)$. The projections of P from the base points upon the base planes will be $P_1=(0, p_2, p_3, p_4)$, $P_2=(p_1, 0, p_3, p_4)$, $P_3=(p_1, p_2, 0, p_4)$, $P_4=(p_1, p_2, p_3, 0)$. The following are the vertices of the P-complementary tetrahedron. $P^1=(-p_1, p_2, p_3, p_4)$, $P^2=(p_1, -p_2, p_3, p_4)$, $P^3=(p_1, p_2, -p_3, p_4)$, $P^4=(p_1, p_2, p_3, -p_4)$. The base tetrahedron and the P-complementary tetrahedron are quadruply perspective, with centers

of perspectivity at P, and at $P'=(p_1, p_2, -p_3, -p_4)$, $P''=(p_1, -p_2, p_3, -p_4)$, $P'''=(p_1, -p_2, -p_3, p_4)$. These form a desmic system. The plane containing P', P'', P''', namely

$$\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}=0$$

is called the P-double plane.

The vertices of the base tetrahedron and of its P-complementary tetrahedron together constitute the vertices of the P-hexahedron, whose six face planes are $(x_i/p_i)+(x_j/p_j)=0, j\neq i$. Each of the face planes contains one edge of the base tetrahedron and one edge of the P-complementary tetrahedron.

The three principal planes of the P-hexadron are

$$\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}-\frac{x_4}{p_4}=0, \quad \frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}=0, \quad \frac{x_1}{p_1}-\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}=0.$$

These pass through P. Each contains two of the points P', P'', P'''. Each contains four of the following six points of intersections of edges of the base tetrahedron with edges of the P-complementary tetrahedron: $P_{12}=(0, 0, p_3, p_4)$, $P_{13}=(0, p_2, 0, p_4)$, $P_{23}=(p_1, 0, 0, p_4)$, $P_{14}=(0, p_2, p_3, 0)$, $P_{24}=(p_1, 0, p_3, 0)$, $P_{34}=(p_1, p_2, 0, 0)$. The study of the desmic properties may be based upon the identity

$$\left(-\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)$$
$$+\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)$$
$$=16\,x_1x_2x_3x_4/p_1p_2p_3p_4.$$

Analogous identities whose geometric import might be discussed are,

$$\left(\left(\frac{x_1}{p_1}\right)^2-\left(\frac{x_2}{p_2}\right)^2\right)\left(\left(\frac{x_3}{p_3}\right)^2-\left(\frac{x_4}{p_4}\right)^2\right)+\left(\left(\frac{x_1}{p_1}\right)^2-\left(\frac{x_3}{p_3}\right)^2\right)\left(\left(\frac{x_4}{p_4}\right)^2-\left(\frac{x_2}{p_2}\right)^2\right)$$
$$+\left(\left(\frac{x_1}{p_1}\right)^2-\left(\frac{x_4}{p_4}\right)^2\right)\left(\left(\frac{x_2}{p_2}\right)^2-\left(\frac{x_3}{p_3}\right)^2\right)=0,$$

and

$$4\left(\left(\frac{x_1}{p_1}\right)^2+\left(\frac{x_2}{p_2}\right)^2+\left(\frac{x_3}{p_3}\right)^2+\left(\frac{x_4}{p_4}\right)^2\right)$$
$$=\left(-\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2+\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2+\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2$$
$$+\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)^2=\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}+\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2+\left(\frac{x_1}{p_1}+\frac{x_2}{p_2}-\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)^2$$
$$+\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}+\frac{x_3}{p_3}-\frac{x_4}{p_4}\right)^2+\left(\frac{x_1}{p_1}-\frac{x_2}{p_2}-\frac{x_3}{p_3}+\frac{x_4}{p_4}\right)^2.$$

The six diagonal planes of the P-hexahedron are given by $(x_i/p_i)-(x_j/p_j)=0, j\neq i$. Each contain one edge of the base tetrahedron, and one of the P-complementary tetrahedron.

The P-circumquadric $\sum x_ix_j/p_ip_j=0$ passes through the eight vertices of the P-hexahedron. The point P, and the P-double-plane are pole and polar with respect to this quadric.

The three P-edge quadrics

$$\frac{x_1x_2}{p_1p_2}-\frac{x_3x_4}{p_3p_4}=0,\quad \frac{x_1x_3}{p_1p_3}-\frac{x_2x_4}{p_2p_4}=0,\quad \frac{x_1x_4}{p_1p_4}-\frac{x_2x_3}{p_2p_3}=0$$

are such that each contains four edges of the base tetrahedron, and contains two lines joining opposite points among the set P_{ij} and hence also contains P. Each also contains one of the three lines in the P-double-plane joining P', P'', P''' in pairs.

The three P-hexahedral ruled edge quadrics are

$$\frac{x_1x_2}{p_1p_2}+\frac{x_3x_4}{p_3p_4}=0,\quad \frac{x_1x_3}{p_1p_3}+\frac{x_2x_4}{p_2p_4}=0,\quad \frac{x_1x_4}{p_1p_4}+\frac{x_2x_3}{p_2p_3}=0.$$

Each contains two pairs of opposite edges of the base tetrahedron and two pairs of opposite edges of the P-complementary tetrahedron, and two of the three lines $P'P''$, $P'P'''$, $P''P'''$.

The P-pencil of quadrics consists of all quadrics whose equations are expressible in the form

$$m\sum_i x_i^2/p_i^2+n\sum_{i\neq j}x_ix_j/p_ip_j=0.$$

This pencil contains among others

1. The P-double plane counted twice, $m=1$, $n=2$.
2. The P-circumquadric, $m=0$, $n=1$.
3. The degenerate quadric whose only real point is P, $m=3$, $n=-2$.
4. The P-inscribed quadric, inscribed in the base tetrahedron and in the P-complementary tetrahedron, $m=1$, $n=-1$.
5. The quadric touching the six edges of the base tetrahedron and the six edges of the P-complementary tetrahedron, and inscribed in the P-hexahedron, $m=2$, $n=-1$.

For each point $(x_1, x_2, x_3, x_4,)$ not on the face of the base tetrahedron, there is a unique P-reciprocal point, given by

$$\left(\frac{p_1^2}{x_1},\frac{p_2^2}{x_2},\frac{p_3^2}{x_3},\frac{p_4^2}{x_4}\right).$$

The P-circumquadric, the three P-edge quadrics and the three P-hexahedral ruled edge quadrics are self reciprocal under P-reciprocation. The points P, P', P'', P''', P^1, P^2, P^3, P^4, are invariant under P-reciprocation and are the only such invariant points.

§4. It is sometimes convenient to introduce new coordinates determined by P, as follows:

$$2y_0 = \frac{x_1}{p_1} + \frac{x_2}{p_2} + \frac{x_3}{p_3} + \frac{x_4}{p_4}, \qquad 2\frac{x_1}{p_1} = y_0 + y_1 + y_2 + y_3$$

$$2y_1 = \frac{x_1}{p_1} + \frac{x_2}{p_2} - \frac{x_3}{p_3} - \frac{x_4}{p_4}, \qquad 2\frac{x_2}{p_4} = y_0 + y_1 - y_2 - y_3$$

$$2y_2 = \frac{x_1}{p_1} - \frac{x_2}{p_2} + \frac{x_3}{p_3} - \frac{x_4}{p_4}, \qquad 2\frac{x_3}{p_3} = y_0 - y_1 + y_2 - y_3$$

$$2y_3 = \frac{x_1}{p_1} - \frac{x_2}{p_2} - \frac{x_3}{p_3} + \frac{x_4}{p_4}, \qquad 2\frac{x_4}{p_4} = y_0 - y_1 - y_2 + y_3.$$

Then the P-double plane becomes $y_0 = 0$. The base points become $B_1 = (1, 1, 1, 1)$, $B_2 = (1, 1, -1, -1)$, $B_3 = (1, -1, 1, -1)$, $B_4 = (1, -1, -1, 1)$. The vertices of the P-complementary tetrahedron are then in y-coordinates, $P^1 = (-1, 1, 1, 1)$, $P^2 = (1, -1, 1, 1)$, $P^3 = (1, 1, -1, 1)$, $P^4 = (1, 1, 1, -1)$, while $P = (1, 0, 0, 0)$, $P' = (0, 1, 0, 0)$, $P'' = (0, 0, 1, 0)$, $P''' = (0, 0, 0, 1)$. The principal planes become $y_2 = 0$, $y_3 = 0$, $y_4 = 0$. The diagonal planes of the P-hexahedron become $y_1 - y_2 = 0$, $y_2 - y_3 = 0$, $y_3 - y_1 = 0$, $y_1 + y_2 = 0$, $y_2 + y_3 = 0$, $y_3 + y_1 = 0$. The P-circumquadric becomes $y_1^2 + y_2^2 + y_3^2 - 3y_0^2 = 0$. The three P-edge quadrics become respectively, (in the order given)

$$y_0y_1 - y_2y_3 = 0, \quad y_0y_2 - y_1y_3 = 0, \quad y_0y_3 - y_1y_2 = 0.$$

The three P-hexahedral ruled edge quadrics become $y_0^2 + y_1^2 - y_2^2 - y_3^2 = 0$, $y_0^2 - y_1^2 + y_2^2 - y_3^2 = 0$, $y_0^2 - y_1^2 - y_2^2 + y_3^2 = 0$. The P-pencil of quadrics becomes

$$(2m - n)\sum_{i=0}^{3} y_i^2 + 4ny_i^2 = 0.$$

The P-reciprocal of the point (y_0, y_1, y_2, y_3) is

$$[y_0^3 - y_0(y_1^2 + y_2^2 + y_3^2) + 2y_1y_2y_3,\ y_1^3 - y_1(y_0^2 + y_2^2 + y_3^2) + 2y_0y_2y_3,$$
$$y_2^3 - y_2(y_0^2 + y_1^2 + y_3^2) + 2y_0y_1y_3,\ y_3^3 - y_3(y_0^2 + y_1^2 + y_2^2) + 2y_0y_1y_2].$$

§5. The equation of the circumsphere will be of the form

$$\sum_{i \neq j} e_{ij}^2\, x_i\, x_j = 0$$

for a general system of tetrahedral coordinates. If special barycentric coordinates be used so that the centroid is given by $(1, 1, 1, 1)$ then the coefficients e_{ij}^2 may be interpreted as the square of the length of the edge joining B_i to B_j. It is often convenient to associate a direction with each edge so that the directed length B_iB_j is given by e_{ij}. Then $e_{ji} = -e_{ij}$. While the choice of directions might be made at random, a "regular" choice will be said to be any choice for which the following three products are of the same sign: $\beta' = e_{12}e_{34}$, $\beta'' = e_{13}e_{42}$, $\beta''' = e_{14}e_{23}$.

The matrix[2]

$$\begin{pmatrix} 0 & e_{12} & e_{13} & e_{14} \\ e_{21} & 0 & e_{23} & e_{24} \\ e_{31} & e_{32} & 0 & e_{34} \\ e_{41} & e_{42} & e_{43} & 0 \end{pmatrix}$$

is then skew symmetric, with determinant $4\sigma^2$, where $2\sigma=\beta'+\beta''+\beta'''$. The inverse matrix is proportional to the skew symmetric matrix,

$$\begin{pmatrix} 0 & e_{34} & e_{42} & e_{23} \\ e_{43} & 0 & e_{14} & e_{31} \\ e_{24} & e_{41} & 0 & e_{12} \\ e_{32} & e_{13} & e_{21} & 0 \end{pmatrix}.$$

The products of the non-zero elements of the several rows will be denoted as follows: $k_1=e_{34}e_{42}e_{23}$, $k_2=e_{43}e_{14}e_{31}$, $k_3=e_{24}e_{41}e_{12}$, $k_4=e_{32}e_{13}e_{21}$. They will be of the same sign. For instance, $k_1k_2=e_{34}e_{42}e_{23}e_{43}e_{14}e_{31}=e_{34}{}^2e_{13}e_{42}e_{14}e_{23}=e_{34}{}^2\beta''\beta'''$, which is positive. Hence k_1 and k_2 have the same sign.

The equation of the circumsphere may now be written in the form

$$\beta'\left(\frac{x_1x_2}{k_1k_2}+\frac{x_3x_4}{k_3k_4}\right)+\beta''\left(\frac{x_1x_3}{k_1k_3}+\frac{x_2x_4}{k_2k_4}\right)+\beta'''\left(\frac{x_1x_4}{k_1k_4}+\frac{x_2x_3}{k_2k_3}\right)=0.$$

Under real projective transformations one may distinguish between the interior and exterior of the circumsphere. In particular each real point (x_1, x_2, x_3, x_4) for which all the coordinates have the same sign is in that one of the eight tetrahedral regions determined by the base points, which is wholly interior to the circumsphere.

By taking $K=(k_1, k_2, k_3, k_4)$ in place of the general point P, and by using the corresponding y-coordinates, the equation of the circumsphere assumes the form

$$(\sigma-\beta')y_1^2+(\sigma-\beta'')y_2^2+(\sigma-\beta''')y_3^2=\sigma y_0^2.$$

While under real projective transformation in the plane, any nondegenerate real conic is transformable into any other, not all nondegenerate real quadrics are equivalent under real projective transformations of space. Since the plane $y_0=0$ does not cut the circumsphere in real points, and yet there are real points upon this circumsphere, we may conclude that not only are β', β'', β''', of the same sign, but σ, $\sigma-\beta'$, $\sigma-\beta''$, $\sigma-\beta'''$, are also of this same sign. It may be noted that

$$[\sigma(\sigma-\beta')(\sigma-\beta'')(\sigma-\beta''')]^{1/2}$$

[2] The rows or columns may be used to identify a tetrahedron inscribed and circumscribed to the base tetrahedron giving a pair of so-called Möbius tetrahedra.

which is analogous to the formula for the area of a triangle in terms of its edges, gives in the metrical interpretation of e_{ij} as the edge B_iB_j, six times the product of the volume of the base tetrahedron by the radius of its circumsphere.

The point (k_1, k_2, k_3, k_4) will be called the Lemoine center, K, of the tetrahedron. One has, as for the general point, P, the associated points, K^1, K^2, K^3, K^4, K', K'', K''', and the Lemoine double plane

$$\frac{x_1}{k_1}+\frac{x_2}{k_2}+\frac{x_3}{k_3}+\frac{x_4}{k_4}=0$$

and so forth.

The Lemoine center and Lemoine double-plane are pole and polar with respect to the circumsphere, which serves to characterize the Lemoine system.

§6. To show directly that the point K is projectively identified, it suffices to show that the set of four points, K^1, K^2, K^3, K^4, is projectively determined. This set is quadruply perspective with the base points. Three of the perspective centers namely K', K'', K''', lie in an exterior plane, polar to the interior point, K, which for real projective purpose is thus specialized.

In each face of the base tetrahedron there is a symmedian center. Thus in $x_1=0$, there is a triangle with sides given by $x_2=0$, $x_3=0$, $x_4=0$, and with circumcircle given by

$$e_{23}^2x_2x_3+e_{24}^2x_2x_4+e_{34}^2x_3x_4=0.$$

The symmedian center is $(0, e_{34}^2, e_{24}^2, e_{23}^2)$. The four lines each joining a base point to the symmedian center of the opposite face lie in the ruled quadric

$$(\beta''^2-\beta'''^2)(e_{12}^2x_1x_2+e_{34}^2x_3x_4)+(\beta'''^2-\beta'^2)(e_{13}^2x_1x_3+e_{24}^2x_2x_4)$$
$$+(\beta'^2-\beta''^2)(e_{14}^2x_1x_4+e_{23}^2x_2x_3)=0.$$

In each face there is a Lemoine axis. Thus in $x_1=0$ the Lemoine axis is given by

$$e_{23}^2e_{24}^2x_2+e_{23}^2e_{34}^2x_3+e_{24}^2e_{34}^2x_4=0.$$

This meets the line of intersection with $x_1=0$ of the plane tangent to the circumsphere at B_1, namely the plane $e_{12}^2x_2+e_{13}^2x_3+e_{14}^2x_4=0$, in the point

$$[0, e_{34}^2(\beta''^2-\beta'''^2), e_{24}^2(\beta'''^2-\beta_1^2), e_{23}^2(\beta'^2-\beta''^2)].$$

The line joining this point to the base point B_1 is one of four such lines all of which lie on the ruled quadric.

$$(2\beta''^2\beta'''^2-\beta'^2\beta''^2-\beta'^2\beta'''^2)(\beta''^2-\beta'''^2)(e_{12}^2x_1x_2+e_{34}^2x_3x_4)$$
$$+(2\beta'^2\beta'''^2-\beta'^2\beta''^2-\beta''^2\beta'''^2)(\beta'''^2-\beta'^2)(e_{13}^2x_1x_3+e_{24}^2x_2x_4)$$
$$+(2\beta'^2\beta''^2-\beta'^2\beta'''^2-\beta''^2\beta'''^2)(\beta'^2-\beta''^2)(e_{14}^2x_1x_4+e_{23}^2x_2x_3)=0.$$

These two ruled quadrics meet the circumsphere in a self-associated set of eight points namely the vertices of the Lemoine hexahedron, which latter is thus projectively determined. As in the case of the triangle in the plane, it is convenient to introduce a special Euclidean metric not related to the metric used for the study of other properties of the tetrahedron, but merely used to make available in familiar language projective relations.

The Lemoine double plane, $y_0=0$, will be taken as the plane at infinity. The Lemoine hexahedron is now the cube with vertices $(1, \pm 1, \pm 1, \pm 1)$. The Lemoine principal planes are the finite coordinate planes, $y_1=0$, $y_2=0$, $y_3=0$. The Lemoine circumquadric becomes the Euclidean sphere $y_1^2+y_2^2+y_3^2=3y_0^2$, while the original circumsphere becomes interpreted as an ellipsoid whose three (in general unequal) axes, are the finite coordinate axes.

§7. In connection with the special metric, the notion of three mutually orthogonal planes, describes what may be called a Lemoine trihedron, abstractly defined as a trihedron which meets the Lemoine double plane in a triangle self-polar with respect to the imaginary conic in which the Lemoine circumquadric meets this plane. Numerous examples of Lemoine trihedra present themselves. Thus one has in particular

1. The set of the three Lemoine principal planes.

2. The three face planes of the Lemoine hexahedron at any one of its vertices.

3. Two Lemoine principal planes and the plane tangent to the circumsphere at its intersection with these principal planes.

4. Two Lemoine principal planes and the plane tangent to the Lemoine circumquadric at its intersection with these principal planes.

5. Three tangent planes (no two intersecting the Lemoine double plane in a common line) to the circumsphere at its intersections with the Lemoine axes.

6. Three tangent planes (no two intersecting the Lemoine double plane in a common line) to the Lemoine circumquadric at its intersections with the Lemoine axes.

It is known that the locus of the vertex of a variable trirectangular trihedron tangent to an ellipsoid is a sphere concentric with it. Hence the locus of the vertex of a Lemoine trihedron tangent to the circumsphere is a quadric in the Lemoine pencil.

Interpreting the facts concerning a system of confocal quadrics, we conclude that there are two quadrics which in the special metric sense are confocal with the initial circumsphere and passing through the eight vertices of the Lemoine hexahedron and whose tangent planes at each of the eight vertices of this hexahedron form Lemoine trihedra.

§8. The correspondence

$$z_1 = 1 \Big/ \left(\frac{x_1x_2}{k_1k_2}+\frac{x_3x_4}{k_3k_4}\right),\quad z_2 = 1 \Big/ \left(\frac{x_1x_3}{k_1k_3}+\frac{x_2x_4}{k_2k_4}\right),\quad z_3 = 1 \Big/ \left(\frac{x_1x_4}{k_1k_4}+\frac{x_2x_3}{k_2k_3}\right)$$

carries the Lemoine circumquadric into $z_2z_3+z_3z_1+z_1z_2=0$, and the circumsphere into $\beta'z_2z_3+\beta''z_3z_1+\beta'''z_1z_2=0$.

A linear equation in the z's corresponds to a quartic surface. This quartic has the special properties of meeting each face of the base tetrahedron in four lines of which three constitute the base triangle. Thus in $x_4=0$, the equation $a_1z_1+a_2z_2+a_3z_3=0$, reduces to $a_1k_2k_3x_1+a_2k_3k_1x_2+a_3k_1k_2x_3=0$ save for the extraneous factors.

A quadratic equation in z_1, z_2, z_3, without squared terms corresponds to a quadratic in x_1, x_2, x_3, x_4 without squared terms. Thus a line in the z-plane corresponds to a line in each face of the base tetrahedron. A conic circumscribed about the reference triangle in the z-plane corresponds to a conic circumscribed about the reference triangle in each of the faces of the base tetrahedron. A point in the z-plane corresponds save for extraneous fixed points to a point in each face of the base triangle. When reference is made to the whole of space, a point in the z-plane corresponds to a space quartic through the eight vertices of the Lemoine hexahedron. One may interpret each element in the metric geometry of the triangle as an element projectively determined in the geometry of the tetrahedron, where points correspond to quartic curves. For the special point (1, 1, 1) in the z-plane, the quartic reduces to the four lines joining K to the respective base points.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

QUESTION CONCERNING THE MAXIMUM TERM IN THE DIATOMIC SERIES

By ALBERT A. BENNETT, Brown University

Given k, let π_k denote the continued product of the first k primes, p_i, starting with $p_1=2$. For example, $\pi_5=2\cdot3\cdot5\cdot7\cdot11=2310$. Let π_0 be 1.

Question: *Given k, what is the maximum number of consecutive integers each of which is divisible by one of the primes $p_1, p_2, \cdots, p_k$?*

It is necessary to consider only the sequence of residues modulo π_k. The question asks for the maximum interval between adjacent entries in the sequence of all those numbers which are relatively prime to π_k. Otherwise stated it asks for the maximum term in the diatomic series[1] for p_k.

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[1] These periodic series were introduced by de Polignac in 1851 in partial application of the Sieve of Eratosthenes. A description with references is given in L. E. Dickson, *History of the Theory of Numbers*, vol. 1, (1919), p. 439. Among further properties, one may readily show that for $k>1$, in any period the number of ones is equal to the number of threes, $=(p_2-2)(p_3-2)\cdots(p_k-2)$, while each other odd number that occurs, appears an even number of times in a period.

It is obvious that a sequence of at least $p_{k+1}-2$ successive integers of the sort desired is always available since 2, 3, 4, 5, 6, $\cdots$, $p_{k+1}-1$, form such a sequence, and this type of sequence is maximal for $k=1, 2, 3$. But it is also true that a sequence of $2p_{k-1}-1$ (ordinarily greater than $p_{k+1}-2$) is always available for $k>1$. Indeed let x be a positive solution of the three simultaneous congruences:

$$x \equiv 0 \pmod{\pi_{k-2}},\ x \equiv \pm 1 \pmod{p_{k-1}},\ x \equiv \mp 1 \pmod{p_k},$$

then $x+i$, $(-p_{k-1}<i<p_{k-1})$ denotes a sequence of $2p_{k-1}-1$ numbers each of which is divisible by some factor of π_k. Is there ever (for k given) a longer sequence? For example, for $k=5$ the sequence of thirteen consecutive numbers, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, is such that each is divisible by one of the first five primes, 2, 3, 5, 7, 11, and is a maximal sequence.

NOTE: If it can be proved that for each $k>1$, no sequence of $2p_k+2$ consecutive integers exists each of which is divisible by at least one factor of π_k, it would also hold that no sequence of $2m+2$ consecutive integers exists each of which is divisible by some integer not exceeding m, and hence, as one result, in the sequence $m^2-1, m^2, m^2+1, \cdots, m^2+2m$, at least one term could not contain a factor less than $m+1$. Since $m+1$ exceeds the square root of m^2+2m, the existence of at least one prime number between m^2 and $(m+1)^2$ would then be established (one of those numerous, plausible, and "asymptotically true," but as yet unproven relations!).

A CORRECTION

By MORGAN WARD, California Institute of Technology

Mr. Hans Rohrbach of the University of Berlin has kindly called my attention to an error in a paper of mine in the Monthly (A Simplification of Certain Problems in Arithmetical Division, Vol. XXXV, No. 1, Jan. 1928) which I should like to correct here.

On page 11 of the paper cited, the last summation in the last line should read

$$\sum_{\sigma_1=1}^{r-1-\alpha} a^{\alpha+\sigma_1} V_{r-\sigma_1}$$

instead of

$$\sum_{\sigma_1=1}^{r-1-\alpha} a^{\alpha+\sigma_1} V_{\alpha+\sigma_1}.$$

Also on page 12, the last line of Formula IV should read

$$\frac{a^{kr}-1}{a^r-1}(a^{r-1}V_{\alpha+1} + a^{r-2}V_{\alpha+2} + \cdots + a^{\alpha+1}V_{r-1})$$

instead of

$$\frac{a^{kr}-1}{a^r-1}\, a^{\alpha}(aV_{\alpha+1} + a^2V_{\alpha+2} + \cdots + a^{r-\alpha-1}V_{r-1}).$$

These corrections entail changes in the numerical examples; for instance, we have

$$\left[\frac{2^{257}-1}{1023}\right] = \frac{2^{250}-1}{2^{10}-1}\cdot 2^7 = 2^7 + 2^{17} + 2^{27} + \cdots + 2^{247}.$$

A Simple Geometrical Paradox—Proposed by J. L. Coolidge[1]

Replies by Grace Bareis and Vladimir F. Ivanoff

The Paradox

Question: A point P in three-dimensional Euclidean space is required to be collinear with two given points A and B; how many independent algebraic conditions are thereby imposed upon its coordinates?

First answer: In order to be collinear with A and B it is necessary and sufficient that P should be coplanar with A and B and each of two arbitrary points not in the same plane through A and B; *two conditions.*

Second answer: If P be collinear with A and B it is necessary and sufficient that the sum of two of the distances determined by two pairs from these three points should be equal to the distance determined by the third pair: hence but one condition is imposed.

Which answer is right? How many conditions are imposed in a Euclidean space of n dimensions?

1. *A Reply by Grace Bareis, Ohio State University.*

That the first answer is the correct one is easily seen by making a proper choice of axes, which in no way affects the generality of the conclusion.

We select A as origin and AB as x-axis; then, using the points

$$A(0, 0, 0),\ B(b, 0, 0),\ P(x, y, z),$$

the condition

$$AB^4 + BP^4 + PA^4 - 2BP^2\cdot PA^2 - 2PA^2\cdot AB^2 - 2AB^2\cdot BP^2 = 0$$

reduces at once to $y^2+z^2=0$, or $y=0$ and $z=0$ for real points. Therefore two conditions are imposed upon the coordinates of P.

Similarly, in Euclidean space of n dimensions, we may choose

$$A(0, 0, 0, \cdots, 0,)B(b, 0, 0, \cdots, 0)P(x_1, x_2, x_3, \cdots, x_n).$$

The above condition then becomes

$$x_2^2 + x_3^2 + \cdots + x_n^2 = 0, \text{ or } x_2 = 0, x_3 = 0, \cdots x_n = 0.$$

Therefore $n-1$ conditions are imposed upon the coordinates of the point P if it is required to be collinear with the two given points A and B.

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[1] This paradox was proposed by Professor Coolidge in the April issue of this Monthly, vol. 38 (1931) p. 22.

2. *A Reply by Vladimir F. Ivanoff, San Francisco, California.*

The paradox proposed by Professor Coolidge may be explained in the following way.

We take

$$PA^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = a_{11}^2 + a_{12}^2 + a_{13}^2,$$
$$BP^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = a_{21}^2 + a_{22}^2 + a_{23}^2,$$
$$AB^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 = a_{31}^2 + a_{32}^2 + a_{33}^2,$$

whence

$$(1) \qquad a_{11} + a_{21} + a_{31} = 0,\ a_{12} + a_{22} + a_{32} = 0,\ a_{13} + a_{23} + a_{33} = 0,$$

no matter whether P lies on AB or not.

The condition

$$AB^4 + BP^4 + PA^4 - 2BP^2 \cdot PA^2 - 2PA^2 \cdot AB^2 - 2AB^2 \cdot BP^2 = 0$$

takes the following form:

$$(2) \qquad \sum_1^3 a_{ij}^4 + \sum_1^3{}_{(j \neq k)} a_{ij}^2 a_{ik}^2 - \sum_1^3{}_{(i \neq k)} a_{ij}^2 a_{kl}^2 = 0.$$

From (1) we have

$$(a_{1j} + a_{2j})^2 = a_{3j}^2 \text{ or } a_{1j}^2 + a_{2j}^2 - a_{3j}^2 = -2a_{1j}a_{2j};$$

and squaring both sides of the last expression,

$$(3) \qquad a_{1j}^4 + a_{2j}^4 + a_{3j}^4 - 2a_{1j}^2 a_{2j}^2 - 2a_{2j}^2 a_{3j}^2 - 2a_{3j}^2 a_{1j}^2 = 0;\ j = 1, 2, 3.$$

Again, using (1) we may prove the following identities:

$$(4) \qquad \begin{aligned} &a_{1i}^2 a_{1j}^2 + a_{2i}^2 a_{2j}^2 + a_{3i}^2 a_{3j}^2 - a_{1i}^2 a_{2j}^2 - a_{2i}^2 a_{1j}^2 \\ &\quad = 2a_{1i}a_{1j}a_{3i}a_{3j} + 2a_{2i}a_{2j}a_{3i}a_{3j}; \qquad i \neq j \end{aligned}$$

Taking into account (3) and (4) we get from (2):

$$(a_{11}a_{32} - a_{12}a_{31})^2 + (a_{12}a_{33} - a_{13}a_{32})^2 + (a_{13}a_{31} - a_{11}a_{33})^2$$
$$+ (a_{21}a_{32} - a_{22}a_{31})^2 + (a_{22}a_{33} - a_{23}a_{32})^2 + (a_{23}a_{31} - a_{21}a_{33})^2 = 0,$$

which gives us six conditions, only two being independent. As such we may take

$$(x_1 - x_2)/(x_3 - x_1) = (y_1 - y_2)/(y_3 - y_1) = (z_1 - z_2)/(z_3 - z_1).$$

It is evident that in n-dimensional space we have $(n-1)$ conditions.

Replies were also received from J. Rosenbaum and W. R. Ransom.

A Note by the Editor

This "paradox" devised by Professor Coolidge emphasizes an important point, undoubtedly well known but often overlooked, namely, that if one limits the discussion to real functions of real variables no significance can be attached to the process of counting analytic conditions imposed on them. For under these restrictions of reality, any geometric configuration which can be characterized by a *set of several* analytic equations can also always be completely characterized by *just one* analytic equation, whether the configuration be in two, three or n-dimensional Euclidean space or indeed in any metric space of a denumerably infinite number of dimensions. It is very easy to see that this is the case, for suppose that

$$f_i(x_1, x_2, \cdots, x_m, \cdots) = 0 \qquad i, m = 1, 2, 3, \cdots$$

be the set of analytic conditions.

This set is equivalent to the single equation

$$\sum_i [f_i(x_1, \cdots, x_m, \cdots)]^2 = 0,$$

where the f's and x's are real.

And thus the explanation of Professor Coolidge's paradox may be said to be that his second formulation is valid only in real Euclidean three space while the first is equally valid in real or complex space.

R.E.G.

RECENT PUBLICATIONS

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All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Humanism and Science. By Cassius Jackson Keyser. New York, Columbia University Press, 1931. xx+243 pages. $3.00.

The logical structure of this book is exceedingly simple and apparent. If one disagrees with its thesis it will be very easy to put his finger on the exact spot where the disagreement occurs. After reading "literary" and "philosophical" productions dealing with "humanism," weltering, as many of them do, in vague words and confused sequences of statements and paragraphs, it is a relief as from a nightmare to come upon this clear and beautiful exposition.

The book is in Professor Keyser's well known style, a style that certainly is in great contrast to the indignities upon our language that are all too common in the writings of specialists in the sciences. There is moreover much good poetry, though it is written in the form of prose.

The contents of the book can be put before the reader most effectively by quoting its own words, and that is done freely in this review.

In the Foreword we read:

"Somebody ought to tell, or try to tell, the truth about the relations of Humanism to Science.

"In the extensive literature that the so-called new 'humanistic movement' has recently produced in England, in France and especially in America, the relations of Humanism— or what the writers have called Humanism—to literary criticism, to the criticism of art, to education, to politics, to psychology, to philosophy, to ethics, to religion, and to other cardinal interests, have been pretty fully discussed, but the task of setting forth its relations to the most momentous and characteristic intellectual enterprise of the modern world has not even been seriously undertaken, much less performed. And so I have decided to try it myself."

In the literature referred to there are frequent indications that Humanism and Science are believed to be antagonistic—that this movement is directly at variance with the spirit and the aims of what Keyser rightly calls the "most momentous and characteristic intellectual enterprise of the modern world." The main thesis of Professor Keyser's book is a denial of this belief. To maintain this thesis it is necessary to inquire into the exact meaning of the terms "Humanism" and "Science," and this Professor Keyser at once proceeds to do. What meaning do those who regard themselves as identified with this movement attach to the word humanism? Professor Irving Babbitt, who has been said to be at the center of the humanist movement, says: "Humanists are those who, in any age, aim at proportionateness through a cultivation of the law of measure." This meaning of the word, "when understood in accordance with its historical significance and its philosophical implications," Professor Keyser finds far too narrow.

"In the light of the foregoing considerations it is, I think, abundantly evident that the 'strict' doctrine of Professor Babbitt and his kind is too narrow, too sectarian, too prim, too supercilious, too dogmatic, too pharisaical, too lacking in catholicity, in sympathy, and in magnanimity, to bear worthily the historically great name Humanism."

Other meanings currently assigned to humanism are examined and then Professor Keyser says:

"For brevity, penetration, clarity and comprehensiveness the best description I have seen is in the following words of Mr. Walter Lippmann: 'Humanism signifies the intention of men to concern themselves with the discovery of a good life on this planet by the use of human faculties.' What is here so compendiously described is no cult, no doctrine, no sect, no dogma, but a human 'intention.' Not the intention of depraved men to escape, by humble submission to ecclesiastical authority and by means of divine grace, from the eternal wrath of an angry God; not the intention of superstitious men to propitiate, by purchase or prayer

or sacrifice, the countless malignant spirits of an imaginary environment; not the intention of dupes to submit supinely to the indignities of the present world in the hope of everlasting bliss in a world beyond the grave; but the intention of perfectly natural men 'to concern themselves with the discovery'—or as I should prefer to say, with the creation—'of a good life' here upon this mundane sphere where they actually live, and to do it by the application of native powers resident in themselves to the waiting resources of the actual world."

"In respect of spirit and aim and implicit scope the conception thus indicated accords perfectly with the Humanism of antiquity · · · And it accords completely with the Humanism which, after the lapse of more than a thousand years including the Dark Ages, began in the fourteenth century to emerge again, sprang into full life in the fifteenth century and greatly flourished as an essential and momentous part of the Rennaissance first in Italy and later in other countries of Europe. Humans are, as human, naturally endowed with the dignity of autonomous beings, potentially qualified by native inheritance to judge, individually and independently for themselves, in all the great matters of human concern, and, by the exercise of their own faculties, to order and fashion their lives worthily."

"The living sense of personal autonomy is absolutely essential to the dignity of man as man. Whenever and wherever it has been dormant or dead, Humanism has declined or perished. It is from the living sense in men of their personal autonomy that Humanism derives its existence, its character and its power. The fact was continuously manifest in the activity of the great humanists of the Renaissance and frequently became articulate in their words."

So much for the meaning of "Humanism."

The definition of "Science" used by Professor Keyser is that developed in his recent book "Pastures of Wonder," which was reviewed in this journal, vol. 37, p. 81, (see also "Review of a Review," this journal, vol. 38, p. 39).

"Science is the enterprise having for its aim to establish Categorical propositions; in other words, it is the enterprise having for its aim to answer questions relating to the Actual world.

"According to that definition the scope of Science excludes no subject-matter of the Actual world but includes any element thereof, whether material or mental, physical or psychical, with respect to which one can significantly say categorically that it has, or that it has not, such-and-such properties or relations."

Professor Keyser's distinction between "Science" and "Mathematics," which has aroused some controversy[1] is not essential in the discussion of this book, but the extension of the term "Science" to cover a much larger field than the so-called "experimental natural sciences" greatly strengthens his main thesis.

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[1] After some reflection I now wish to record my belief that Professor Keyser's definitions of the words "science" and "mathematics" as given in his "Pastures of Wonder" will be adopted increasingly by those who use these words with discrimination.

It is then maintained that the spirit of science is in the closest accord with the spirit of humanism, and that the achievements of science are the most important means whereby humanism in modern times is moving towards its goal. The argument runs smoothly and irresistibly.

"Science springs from humanistic soil. By that I mean that Science has its base and rootage in human interests, propensities and capacities that, save for variations in degree, are common to all mankind. · · · The spirit of Science—the spirit that seeks Truth, the spirit that leads to wisdom, knowledge, understanding of the Actual—is the animating principle of Man, and it is the soul of Humanism."

Regarding the achievement which science has already made in furthering the cause of humanism we read:

"Here it must suffice to remind ourselves in the most general terms of the immense transformations in our conception of the world and of the countless contributions to the art of life that scientific inventions and other applications of Science have made. It is sufficient to remind ourselves of the fact that such inventions and applications, by conquering not only the continents but also the oceans and the air, and by multiplying many million times our human powers of locomotion, hearing, vision, and speech, thus abolishing what were once insuperable difficulties both of time and of space, have resulted in virtually reducing the entire modern world to the dimensions of an ancient province, thereby bringing the divers people of the globe more and more into the mutual relationship of so many families of one vast community. Who can be so blinded by what remains to be done as not to see that, owing to the agency of Science, immeasurable strides have been taken forward on the endless path towards the realization of Humanism's dream? It is indeed a far cry from a planet inhabited by head-hunting tribes to a world of great nations assembled in friendly council to promote the achievement of a good life for all mankind by the use of human faculties."

Certain difficulties do arise in Professor Keysor's argument: Humanism is said to postulate the "autonomy" of man. This implies freedom of choice, of will. Does this exclude from the ranks of humanists those who hold deterministic views of nature, including man—those who do not believe in the freedom of the will?

Humanists, we are told, seek their ends here on earth and by use of human faculties. Does this exclude those who seek the same ends, in part, by divine help, and does it exclude those who make part of their purpose so to live that they will be placed more advantageously in a future life, or may these be classed as humanists plus something besides?

Does modern science disclose a universe so vast in space and time as to leave man an utterly insignificant mote and his life a flash of movement, "A lantern between dark and dark," and thus lead to despair?

To learn Professor Keyser's way of dealing with these questions one must read his book.

The last chapter of the book is entitled "The Humanistic Bearings of Mathematics." Every mathematician should read this chapter. Frankly, I know of no printed statement which in the space of fifty-odd short pages says so much (and says it so well and so simply) about the essential nature of mathematics and its general human significance. An attempt to epitomize this chapter would be tedious and superfluous.

Some have questioned whether a mathematician should write a book such as this. My answer is emphatically, yes; yes, if he has the wit and taste and information that are necessary to do it. The counter question we may put, why should he not do it? Why should one who has spent a long life in close contact with mathematics and science refrain from a study of the relations of these great fields to more general human interest? Is it not a worthy enterprise to attempt to understand their signifigance as a part of modern life and culture, when these are viewed as a whole? Is it not worth while to attempt to broaden the understanding of specialists in other fields so that instead of being antagonistic to the work of their colleagues and fellows in arms in the common assault on the great unknown, they may understand and appreciate and make common cause? May it not be that the weakest element in the intellectual life of our universities and research institutions is a tendency to fail to evaluate what other so-called "departments" are doing—a tendency not to question the more general significance of special results? A few have disregarded the traditional arrangement into "compartments of thought." In America Professor Keyser is conspicuous among these. England has her Bertrand Russell and her A. N. Whitehead (now of Harvard). In our time, which is supposed to have brought the widest freedom, a freedom so great as to permit even interdepartmental migration of workers, censure for such men is certainly not justified, and if new light is the result of their labors our thanks and appreciation are deserved.

N. J. LENNES

Mathematics for Junior High School Teachers. By William L. Schaaf. Richmond, Johnson Publishing Company, 1931. xiv+440 pages.

This book fills a want that certainly has long been indicated under the curricular arrangement of our present educational system. The reviewer has, till now, read nothing that gives to the candidate, as well as to the present occupant of the position of mathematics teacher so concise and complete a background of essential content.

At the same time practical methods of presentation are not neglected, and the wealth of material for pedagogical study gives the reader plenty of scope to exercise his ability in the direction of original work in this field.

The author's arrangement, which follows a natural tendency of scholastic development along mathematical lines, is logical. He starts with direct measurement, a practical subject which can be checked by simple appeal to the senses; follows this with a chapter on intuitive geometry, goes on to geometrical men-

suration, and then takes up demonstrative geometry and numerical trigonometry. This is followed by a chapter on the statistical graph, emphasizing its value in the field of pedagogy. Then come chapters on arithmetic which in the beginning deal with the common matters that enter a child's life. Subsequently the author analyzes the relations of arithmetic to community business, including banking, investments, and insurance; things with which the reader is bound to come in contact some time or other.

Much attention is given to arithmetic, the attempt obviously being made to give the prospective teacher a very complete survey of this subject and also to prepare the way for algebraic analysis.

The teacher who enters a class room with the background obtained from intelligent use of this book, ought never be at a loss for a practical illustration of the principles he wishes to develop.

The directed number, the formula and the general number, are subject matter for the next chapters and these form a natural introduction to formal algebra and the algebraic graph.

The book is full of meat; there is no waste material in its 400 pages, and as a reference book it should prove invaluable long after it has been thoroughly studied. It seems almost impossible, in a short review, to do it justice; for the table of contents, alone, five pages of condensed printing, gives only the most cursory index of the material summarized.

A course for teachers of mathematics, with this as a text book, should without doubt prove most valuable; and this applies to the aspirant for a collegiate chair almost as much as it does to the Junior High School applicant. To do it justice, however, such a course should be carefully digested and not be hurriedly skimmed over. The questions for discussion at the end of each chapter should be fertile material for making the students think over and develop the modern attitude toward content in mathematics as well as toward practice and theory of teaching.

However, the reader should be warned that this book is one to be *studied*, and does not lend itself to casual reading. He must be prepared to give it undivided and concentrated attention.

From the standpoint of manufacture, like all other publications of the Johnson Publishing Company, it has the merits of being well-bound, well-printed and seems to have been carefully proof-read and edited. A somewhat wider margin might add to the appearance of its pages, and at the same time give opportunity for marginal notes.

ARTHUR HAAS

The Dynamic Universe. By James Mackaye. New York, Charles Scribner's Sons, 1931. 308 pages. $3.50.

The ability and integrity of Einstein have been freely recognized by the early opponents of his theory. The book under review, however, gives the im-

pression of impugning the character of the inventor of the relativity theory. According to this work Einstein is merely adorning himself with borrowed plumes and has succeeded in duping all scientists through "mathematical fictions manipulated at his option." He is even charged overtly with juggling. "Einstein has succeeded, somehow, in conjuring explanations out of the hat by simply juggling definitions and units." He is accused of appropriating ideas of others and putting them forth as his own under some disguise. "Einstein's theory is only a disguise · · ·. The relativity equations are disguises for classical laws." The character of one guilty of juggling and abstracting other peoples' ideas would appear to be anything but honorable.

The relativity theory is utterly destroyed in *The Dynamic Universe.* It is proved to be wrong in general and, what is even worse, to be deliberately deceptive, plagiaristic. "The relativity equations are everywhere disguises." It is claimed that the theory has been harmful to physical research through the widespread confusion which it has produced in the realm of physics.

This general picture of the relativity theory given in the book is unfair, to say the least. A theory which has been fully approved by the best physicists of our age, such as Planck, Laue, Michelson, etc., and whose originator has been awarded the Nobel prize, cannot be as worthless as it appears to be from the book. This argument, however, of wide recognition of the theory is not sufficient to prove its criticism to be wrong. For this the book furnishes a far stronger argument, namely, evidence of a considerable lack of knowledge of the theory on the part of the author.

The book condemns the relativity theory on the ground that it is based on definitions: "Einstein begs the question of whether time, length, etc. are relative by defining them to be so. · · · Time and length are relative by definition." Such passages reveal misunderstanding of the relativity theory. Time and length are relative in it, not by "defining them to be so," but as a consequence of the two fundamental principles or assumptions of the theory. The validity of these being granted, it can be proved conclusively that simultaneity of distant events, an interval of time, and a spatial length are different for different observers. These factors are, therefore, relative not "by definition" but by proof deduced from two principles.

That failure to understand the relativity theory underlies its criticism in the book may be seen from the following citations. The assertion is attributed to Einstein that "a gravitational field can be produced by somebody pulling on a rope" or that "pulling on a rope produces a gravitational field." This is denied with the remark that "neither pulling on ropes, nor any other means of accelerating bodies can alter the gravitational field of the earth." To put such an interpretation on Einstein's famous illustration of his equivalence principle betrays insufficient acquaintance with the relativity theory. Einstein asks us to imagine in the interstellar space a room pulled "upwards" with constant force by some being. The behavior of all objects inside the room would in nowise be distinguishable from that which they would show if the room, instead of being pulled

"upwards," were in a gravitational field acting "downwards." In both cases an observer in the room would see that objects which he lets loose approach the floor with an acceleration which is the same for all of them, the same for a small pellet of cork as for a big lump of lead. This means that acceleration in one direction through any force is equal in its observable effects to acceleration in the opposite direction through gravitational force. Only misunderstanding of the relativity theory can see in that lucid illustration the claim that "pulling on ropes can alter the gravitational field of the earth."

Further proof of miscomprehension of the relativity theory is furnished by the book through the following argument. The general relativity theory is founded upon the equality of gravitational mass and inertial mass. The book tries to destroy this foundation by proving that gravitation and inertia are dissimilar. It resorts to an imaginary experiment. The weight and inertia of a body are measured at the Equator. The body is then removed to the Pole and the measurements are repeated. They show that the weight of the body has increased but its inertia has remained the same. From this the conclusion is drawn that there is "a fundamental dissimilarity" between gravitation and inertia. Hence the basis of the general theory is destroyed. But the conclusion contains a grave error. From the experiment one may infer only the inequality of weight and inertia but not that of gravitational mass and inertial mass. That conclusion shows that gravitational mass has been confounded with weight. But the two are not identical. Gravitational mass is the quotient obtained by dividing weight by gravitational acceleration. This quotient does not vary with the place on the earth. Its value for a body weighing one kilogram in Paris can be shown to be 0.10195 everywhere on the earth. The inertial mass is expressible by the same quotient. Hence gravitational mass and inertial mass are not made unequal by that experiment. They appear to be unequal in the book because of its confounding gravitational mass with weight.

The book disparages the relativity theory for its failure to explain Nature, that is, to reveal cause and inner mechanism of all natural phenomena: "The power of the general relativity theory to explain anything is denied by some of the ablest advocates thereof. . . . The total matter theory can neither explain nor predict anything." With regard to such statements—there are many more of them in the book—the author of the relativity theory may well say, "God protect me from my friends, against my enemies I shall defend myself." The writings of some of the stanchest supporters of the theory have created the impression that it explains everything and anything in the universe; that is, discloses cause and inner workings of all natural phenomena. But the theory pretends nothing of the kind. The special relativity theory makes no claim regarding the explanation of the natural phenomena except that "two assumptions are sufficient to arrive at a simple electroydnamics of moving bodies free from contradiction." No assertion is made that these assumptions give an account of cause and workings of electromagnetism, nor even of light which plays the most important part in the theory. Only the observable optical phenomena are

treated in it, namely, the facts that light is propagated and that it has a certain velocity in a given medium through which it passes. The cause of light and its mode of propagation are left out of the question entirely.

Neither does the general theory pretend to reveal cause and inner workings of the physical phenomena of gravitation and inertia which figure most prominently in it. Nothing is alleged regarding their cause and intrinsic mode of operation except that the gravitational and inertial phenomena observable in a body may have their cause not in the body alone but also in the masses of the universe outside of it. No attempt is made to explain that cause. Censuring the relativity theory for untenable claims not made in the original monographs on it nor in related writings of its creator is unjustifiable.

The author of *The Dynamic Universe* proposes a theory of his own, called the *radiation theory*. With regard to claims it does not compare favorably with the relativity theory. The latter is distinguished through the great moderation of its claims. It does not pretend to explain any natural process, that is, to lay bare its cause and inner mechanism. The new radiation theory, on the other hand, seems to pride itself on being able to explain in this sense all physical phenomena, static electricity, light, attraction, repulsion, impenetrability, gravitation, inertia, etc.

I can say but very little about this new radiation theory because it is unintelligible to me. Whether physicists will understand it and pay attention to it remains to be seen. To me the theory appears to refute itself through the extravagance of its claims. There are a good many vague statements in the presentation of the new theory. I shall merely point out one strange assertion. Discussing "radiation displacement" the book remarks that "it has no more to do with space and time than physical phenomena in general." This can only mean that physical phenomena in general have nothing to do with space and time. Hence they must occur outside of space and time. One may well wonder how a physical phenomenon can occur in no place and at no instant. A radiation displacement having nothing to do with space is a palpable contradiction in itself. A displacement always implies space. But perhaps the "radiation displacement" of the new radiation theory is no displacement at all. What it actually is one cannot gather from the book.

Nevertheless science may indirectly benefit by the book under review. Through the work attention is called to the necessity of purging the relativity theory of dross injected into it by metaphysicians, and physicists with a bent towards mysticism. Their exaggerated claims pass for an integral part of the theory, marring its original beauty and simplicity and rendering it incomprehensible. The mystically inclined physicists would want the theory to be a panacea for all the troubles and difficulties in explaining Nature. In the end the panacea does not work, and then they advocate, as a remedy for those troubles, views akin to the exploded superstitions of the dark ages. Severe criticism is administered to such "relativists" in *The Dynamic Universe* and is fully justified. The relativity theory itself becomes involved in this criticism when their vaga-

ries pass for a part of the theory. Ask a layman what the relativity theory is, and he will probable answer that it is a doctrine asserting that a "fourth dimension" exists and that "space is curved." These ideas have been harped upon too much, and many relativists have only added to their mysteriousness. "Fourth dimension" is one of four straight lines perpendicular to each other in one point, and "curved space" has the same character as a crooked line and a bent surface. Our objection that these two things are impossible is brushed aside by the empty argument that we merely do not possess the power to visualize them. From those mystical notions the inference is drawn that right may be left and left may be right, past may come after future, and effect may precede the cause. Such preposterous conclusions are arrived at through "verbal hocus-pocus," as the book expressively characterizes the unintelligible talk of the relativists.

Einstein used those two terms in a radically different sense. Had he foreseen the confusion they have caused, he would probably have avoided them in expounding his theory. This can be done very well, as I have shown in a paper entitled "*A Simple Presentation of the Fundamentals of the Relativity Theory*" (Scientific Monthly, January, 1932). It shows that by keeping clear of those two expressions an interpretation of the theory gains in lucidity, frees the theory of all mysteriousness and renders it quite comprehensible.

There are other wrong notions about the relativity theory which constitute foreign dross to be eliminated together with the absurdities mentioned above. It is an error, for instance, that the theory has abolished the ether. Einstein himself states that "according to the general relativity theory a space without ether is inconceivable." Further, the rumor that the theory is incomprehensible and only twelve men in the whole world understand it is a myth. The statement in *The Dynamic Universe* that "few physicists pretend to understand it" will not be admitted by any competent physicist. What is true about the rumor is that even highly educated laymen have very little understanding of the theory. One reason for this is to be found in the encroachments upon it by metaphysics and mysticism. The principal reason is that the faculty for good teaching is a rare natural endowment, as rare as the gift of creating choice poetry; and there has not as yet appeared on the scene representing the relativity theory the talented actor who is both a competent physicist and an ideal teacher.

MAX TALMEY

Allgemeine Natürliche Geometrie und Liesche Transformationsgruppen. By G. Kowalewski. Berlin and Leipzig, W. de Gruyter & Co., 1931, 280 pages. RM 15.50.

In 1889, E. Cesàro published in the Rendiconti dei Lincei the first of a series of articles on the intrinsic geometry of curves. The first two chapters of the present work present this theory, for plane and space curves, as it has been developed out of Cesàro's basic ideas. Starting from the equation $\rho = \rho(s)$, which expresses the radius of curvature of a plane curve as a function of the length of arc, he shows how a Cartesian equation of the curve may be derived and how the

properties of the curve, and the loci associated with it, that are invariant under motion may be determined. Similarly, from the intrinsic equations $\kappa=\kappa(s)$, $\tau=\tau(s)$ which express the curvature and torsion of a space curve in terms of s, an analogous study is made of the invariants under motion of space curves.

The group of motions, however, is only one special type of transformation to which we may wish to subject a curve. In the next two chapters, the author generalizes to other types of Lie transformations the essentials of the ideas and methods that he has derived in the first two chapters. This discussion is more difficult to follow, not only because of its greater generality, but also because more advanced mathematical training on the part of the reader is presupposed. In the last chapter, the fundamentals of Lie's theory of transformation groups is explained and connected with the present treatment of the subject.

The author's style is a bit abrupt, but it is stimulating and suggestive. The mode of exposition is interesting, in that the theory is expounded, as far as practicable, by means of discussions of carefully chosen special cases. The reviewer believes that this form of presentation has been used very successfully. Readers who wish to learn how to use this valuable theory as a tool for their own investigations will be well pleased with the author's presentation of the subject.

C. H. SISAM

Mathematical Tables and Formulas. Compiled by Robert D. Carmichael and Edwin R. Smith. Boston, Ginn and Company, 1931. viii+270 pages. $2.00.

The principal tables in this volume are five-place tables of logarithms of numbers, of logarithms of trigonometric functions, and of natural functions. These tables are arranged in columns down the page instead of the usual arrangement across the page, making the tables rather more bulky but more convenient. There are also four-place tables, a table of important constants, a table of functions of multiples of 15°, tables of functions and logarithms of functions of angles expressed in radians, and the usual conversion tables between the circular and the sexagesimal systems. In Part II there are tables of powers, roots, and reciprocals, natural logarithms, exponential and hyperbolic functions, of multiples of $M=0.43429$ and its reciprocal, and finally a table of ten-place logarithms of prime numbers less than 1000. Part III consists of over 50 pages of formulas and theorems from algebra, geometry, trigonometry, analytic geometry and the calculus. There are ten pages of standard graphs, an integral table consisting of 360 formulas, and 50 infinite series. All this material is carefully chosen, and the book should be valuable to mathematicians.

R. A. J.

Electricity and Magnetism, The Mathematical Theory. By Vincent C. Poor. New York, John Wiley & Sons, Inc., 1931. vii+183 pages.

This monograph is substantially an abridged exposition of the classical theory of electricity and magnetism. In the author's own words, "this book should appeal to a very select class of readers, the physicists, the mathemati-

cians and the electrical engineers, who wish to orient themselves quickly in electrical theory." It is true that "... the professional electrical engineer is graduated without being able to read electrical theory. This is a deplorable situation." From my own experience, I can state that much of the future "mathematization" of electrical engineering will consist of gradually incorporating the "electromagnetic theory" as distinguished from the "circuit theory" into the thinking of every electrical engineer. Both theories represent invaluable methods of solving "everyday problems" from now on. For these reasons, we rejoice that the present book is clearly written and regret that there are serious omissions and occasional emphases on comparatively non-important aspects of the theory. An eight page introduction dealing with Vector Analysis is followed by a forty page development of electrostatics. There, the concepts of electric force, displacement, electric potential, and so on are defined but unfortunately the important idea of capacity is hardly mentioned and the fundamental electrostatic problem is not emphatically stated. The explanation of the idea of "displacement" is rather cumbersome; this is due to a somewhat unfortunate choice of experimental evidence leading to this particular concept. The magnetic field due to permanent magnets is the subject matter of the second chapter. In the next chapter, dealing with current electricity, the author develops Maxwell's equations. Although he gives the equations in a perfectly general form, no concrete example is given of the significance of these equations in the study of electric flow in conductors; like many other exponents of the classical theory, the author is interested primarily in electromagnetic waves in dielectrics. Although the choice of the subject matter is comparatively immaterial to the mathematicians, the physicists and engineers would have certainly appreciated at least some account of the "skin effect in cylindrical conductors," or some similar problem as the most elementary instance of the utility of Maxwell's equations in the solution of practical problems. Chapter IV is devoted to the dynamical theory of electric currents or—as it is commonly known among electrical engineers—the circuit theory. Starting with D'Alembert's Principle the author arrives toward the end of the chapter at the differential equation of one-mesh circuit and concludes with its solution. The electron theory and the part played by the special relativity in the theory of electricity and magnetism are the topics reserved for the last chapter. Our general impression is: First, that the book is not likely to suit the tastes of the majority of engineers because no concrete and practically significant problems are either solved or stated for solution; second, that incomplete as it is, the treatment is clear, even if old-fashioned in places, but not always the simplest; that on this account, it is suitable as a text for the classroom use in engineering colleges where it could be supplemented by the instructors.

S. A. SCHELKUNOFF

Storia delle Matematiche. Vol. II. I secoli XVI e XVII. By Gino Loria. Padua, Casa Milani ("Cedam"), 1931. 595 pages. 23 lire.

The first volume of this work appeared in 1929 and dealt chiefly with the

classical period, the middle ages, and the renaissance. In a review of that volume (this *Monthly*, vol. 36, p. 387) the late Professor Cajori paid this just tribute to the author: "Combining readability with the requirements of scientific accuracy is an undertaking in which it would be difficult to surpass the distinguished historian from Genoa." Perhaps the very fact that the Italian word for "history" is the same as that for "story" inclines the writers of that country to make their histories more readable than those of other lands. It was the Italian Libri, although writing in French, who made his history of the mathematics of his people read like a fascinating story.

Professor Loria rightly asserts that the two centuries with which he deals, the 16th and 17th, constitute a period with special characteristics of their own. As he says, the years from 1200 to 1500 were relatively barren in comparison with these two centuries. Indeed, measured by the unit of print space, his giving of only a hundred pages more to the mathematics of human history ending with 1500 (as in vol. I) is justifiable. We feel that we live at present in an era of marvellous progress, and so we do; but that the future historian will rank the 18th and 19th centuries very much higher than the 16th and 17th in the opening of new fields of great importance in the domain of mathematics is doubtful.

The chapters follow in due sequence the fifteen of the first volume, and a statement of their general contents is perhaps the best way of giving a fair idea of the ground covered and the arrangement of the material. Stated as briefly as possible the chapter titles and contents are as follows: XVI, Syncopated algebra in its apogee, Part I, Italy, with special reference to the works of Tartaglia, Cardan, Bombelli, and Benedetti; XVII, Syncopated algebra in its apogee, Part II, transalpine, with special attention to Rudolff, Stifel, Recorde, Nunes, Petri, Stevin, van Roomen, and Viète; XVIII, The influence of Humanism on mathematics, as seen in the editions of the Greek classics, and in the works of Maurolico, Commandino, Benedetti, and Guidobaldo del Monte, with an interesting mention of the dawn of the historical study of mathematics (Ramus and Bernadino Baldi); XIX, Trigonometry and cyclometry in the 16th century, at the hands of Werner, Copernicus, Rheticus, and Pitiscus, but more especially as due to Tycho Brahe, Viète, Snell, and Stevin; the sub-titles include the problem of the quadrature of the circle in the 16th century (Clavius, di Monforte, Cristoforo); XX, Aids to scientific correspondence, periodicals, scientific societies, universities; XXI, First years of a glorious century (the 17th), with special attention to Napier, Bürgi, Cataldi, Galileo, Kepler, d'Aguillon, and Bachet de Méziriac; XXII, the disciples of Galileo, particularly Cavalieri, Torricelli, Viviani, Ricci, and Borelli; XXIII, Gli algebristi della vigilia (who but an Italian could express it so poetically and at the same time so exactly?),—Girard, Harriot, Oughtred, and Hérigone; XXIV, The beginning of modern mathematics,—Descartes and Fermat; XXV, The awakening of pure geometry (The Italian is more poetical,—Risveglio, the French *réveil, réveiller*), Desargues and Pascal; XXVI, The precursors of the infinitesimal calculus,—Vincenzo and Tacquet, Roberval, Wallis, de Sluse, Stefano degli Angeli, Mengoli, N. Mercator, Barrow,

David Gregory, and Lord Brouncker; XXVII, The *intermezzo* in which the geometric field was further explored; XXVIII, the Newton-Leibniz controversy.

The work may be characterized in various ways. In the first place it gives the impression of being well-balanced. Although written by one whose interests might tend chiefly to the Italian contributions, there is no evidence of any prejudice in favor of the achievements of his countrymen. Where he has devoted more than the usual attention to Tartaglia and Cardan, he makes us his debtors by his thorough study of the history of the cubic equation, and by his bibliography of the controversy. There are numerous other cases in which his extensive knowledge of the Italian works has enabled him to call attention to matters that have not been so thoroughly considered by most historians, as for example, in the relation of Stevin to Bombelli in the matter of Decimals (p. 111).

Another distinguishing feature is the revaluation of the works of certain writers usually somewhat neglected, such as Giambattista Benedetti, Petri, Guidobaldo del Monte, Scaliger, Simon du Chesne, and d'Aguillon. On the other hand, it might have been better to have given some or more attention to writers like Juan de Ortega, the Taglientes, Vander Hoecke, Köbel, Lazesio, E. de la Roche, dal Sole, Sfortunati, de Boissière, Fernel, Scheubel, Perez de Moya, Glareanus, Digges, Gillibrand, and Henrion, some of whom, however, contributed so little as hardly to be worth mentioning in such a survey as this. The question also arises as to the desirability of neglecting generally the oriental writers of this period some of whom, like Seki Kōwa in Japan, the Jesuits and their followers in China, and Raganātha in India, made contributions worthy of brief mention at least.

A third item that will be welcomed by all readers is the judicial manner in which the Leibniz-Newton controversy is treated, a controversy which would probably never have become acute had it not been for the stupidity of Faccio di Duillier (Nicolas Fatio de Duillier).

The list of corrections to Volume I, given on pages 593–595, is formidable enough to relieve other writers when they contemplate their own shortcomings. On the other hand it raises the question as to possible errors in Volume II. That they exist will be apparent to any reader, as in *librairy* (p. 21 n), *In artem analyticen* (p. 118), *The Grounds of Arts* (p. 105), *The whestone of witte* (pp. 106, 129), *Pathway to knowledge* (good English but not the original spelling), *Teuby* (for Tenby, p. 105), *praise* (for practise, p. 106), and various others of the same nature, easily detected. It would give quite a false impression of the book to call attention to any others, limited, as generally seems to be the case, to minor typographical errors.

In general it may be said that Professor Loria has, in these two volumes, given the story of mathematics up to the year 1700 with fairness and accuracy and with sufficient completeness for the student beginning his reading in the historical field of the science. Mathematicians will look forward with even more interest to a third volume, relating to the 18th and 19th centuries if, as is doubtless the case, this is to be added to his treatment of the subject.

DAVID EUGENE SMITH

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3521. *Proposed by J. M. West, State College, Pa.*

Show that the equations of the tangents of the circle $x^2+y^2=r^2$, through the external point (a, b) are

$$(ar \pm bt)x + (br \mp at)y = r(a^2 + b^2),$$

where t is the length of the tangents from the point (a, b) to the circle.

3522. *Proposed by A. Galbraith, Ash Grove, Mo.*

To trisect any angle, approximately, with straight edge and compasses only take the vertex of the given angle as a center and with any radius describe the arc AC. Divide the chord AC into six equal parts, the middle point being B and the other two points between B and C being F and G in the order B, F, G, C. On AC produced, locate the points H and K such that $CH=\frac{1}{6}AC$ and $HK=\frac{1}{2}CH$. At F erect a perpendicular to AC intersecting the arc AC in E. With G as a center and a radius EG, describe an arc intersecting the chord AC in M. With K as a center and radius EK describe an arc intersecting AC in N. Locate point P, the mid-point of MN. With E as a center and radius PH, describe an arc intersecting AC in L. With L as a center and a radius EL describe an arc intersecting AC in O. Then CO will be the chord of the trisected arc AC.

A Note by the Editors: This is the most accurate method of trisecting any angle approximately by means of a straight edge and compasses only that has ever come under our observation.

We are publishing it for the double purpose of acquainting our readers with this rather complex though very accurate method and to have some of them determine for what angles the method is absolutely exact. We suggest that those who are interested, take a ruler and compasses and follow out the construction. An ocular proof is thus seen to be very convincing to the non-mathematical mind.

3523. *Proposed by Dewey C. Duncan, University of California.*

Solve completely the equation

$$x^7 + Ax^6 + Bx^4 + Cx^2 + D = 0,$$

A, B, C, D, being unknown constants and given that the equation has a double root and one triple root.

3524. *Proposed by W. H. Echols, University of Virginia.*

At the corners of any equilateral triangle ABC let there be hinged three equilateral triangles ALM, BNO, CPQ of any sizes or positions.

Then will the midpoints of each of the sets of three segments

$$(LQ, OP, MN), \quad (BQ, CN, OP),$$
$$(LB, OA, MN), \quad (AP, CM, LQ)$$

be the corners of an equilateral triangle.

3525. *Proposed by Henry A. Campbell, Omaha, Nebraska.*

A semicircle with center O has AC for the diameter at its base. The radius OC is prolonged to D so that $CD = OC = r$. The semi-circumference is bisected at Z, and on the arc ZC the point Y is taken so that ZY subtends at O an angle of 60°. Let the chord ZY produced cut OD in X; and with X as center and radius XD describe a semicircle on the same side of AD as that of the first one. Let B be any point on the arc AZ, and draw OB. Construct on this side of AD the angle DXM equal to the angle AOB, where M is the intersection of the side XM with the second semicircle. Let N be the projection of M on OD, and draw NB cutting again in T the semicircle at O. Prove or disprove that $NT = r$.

Editorial Note. If NT and r were equal, the angle AOB would be three times the angle ONB; and we know that no such construction will give this result for *any* position of B on the arc AZ. Determine the maximum error in this approximate method for trisecting an angle less than 90°, and show that the error is zero only for 90° and a zero angle.

3526. *Proposed by R. E. Gaines, The University of Richmond.*

If a triangle PQR be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$, and if PQ and PR are tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$, the locus of the pole of QR with respect to the ellipse is identical with the locus of the pole of QR with respect to the hyperbola.

3527. *Proposed by R. E. Gaines, The University of Richmond.*

If a variable chord QR of a conic subtends a right angle at a fixed point P on the conic, the chord QR passes through a fixed point S. If P describes the conic, the locus of S is a conic whose equation differs from the equation of the given conic only in the constant term.

SOLUTIONS

434[1917, 328; 1918, 119]. *Proposed by E. W. Chittenden.*

Evaluate $\int_0^1 f(x)dx$ where

$$f(x) = \sum_{n=1}^{n=\infty} \frac{\operatorname{sgn}(x - x_n)}{2^n}.$$

The function sgn $x = -1, 0, +1$ according as x is negative, zero or positive. The numbers x_n form the series

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{7}{8}, \cdots$$

for which the general formula is $x_n = (2h+1)/2^k$, were k is the greatest integer such that $2^{k-1} \leqq n$ and $h = n - 2^{k-1}$.

Note: This problem is reprinted because of a misprint in the original statement and also because of a lack of definiteness in the definition of the series of values x_n. An inquiry has also been received in regard to the origin of the notation "sgn x." This notation seems to be due to Kronecker; cf. Werke, vol. II, p. 500.

Solution by Herbert A. Meyer, Hanover College.

Let $f_n(x)$ be the nth term of the given series for $f(x)$. For each value of n, $f_n(x)$ assumes the value -2^{-n} in the interval $0 \leqq x < x_n$ and the value $+2^{-n}$ in the interval $x_n < x \leqq 1$. When $x = x_n$, $f_n(x) = 0$. The integral of this term from 0 to 1 is $-2^{-n}x_n + 2^{-n}(1-x_n) = 2^{-n}(1-2x_n)$. The term $f_n(x)$ may be represented geometrically by a broken straight line parallel to the x-axis, and its integral is the area between the x-axis and this broken line. Hence the desired integral is

$$\int_0^1 f(x)dx = \sum_{n=1}^{n=\infty} \left(\frac{1-2x_n}{2^n}\right) = 1 - \sum_{n=1}^{n=\infty} \frac{x_n}{2^{n-1}}.$$

These series are absolutely convergent since x_n is not less than zero or greater than one. The error made in stopping with the rth term of the series of which the general term is $2^{1-n}x_n$ will be considerably less than 2^{1-r}.

Also solved by Ralph P. Agnew.

A Note by Otto Dunkel: The infinite series in the result may be made more rapidly convergent by summing groups of its terms. Thus, for $n = 2^{k-1}+i$, $x_n = (2i+1)2^{-k}$, $i = 0, 1, \cdots, 2^{k-1}-1$, $k = 1, 2, 3, \cdots$. Then

$$2^{1-n}x_n = 2^{1-k-2^{k-1}} \cdot (2i+1)2^{-i}.$$

We now find the sum of the second factors for a fixed k. It is easily verified that

$$\Delta[-(2i+3)2^{1-i}] = (2i+1)2^{-i},$$

and hence

$$\sum_{i=0}^{p} (2i+1)2^{-i} = 6 - (2p+5)2^{-p}, \quad p = 2^{k-1} - 1.$$

Hence the result may be written

$$1 - 4\sum_{k=1}^{\infty} [3 \cdot 2^{-k-2^{k-1}} - 2^{-2^k} - 3 \cdot 2^{-k-2^k}].$$

Using five terms, we find its value to 10 decimals to be .1123714457.

3140[1925, 315]. *Proposed by C. C. MacDuffee, Ohio State University.*

Let f be any algebraic form of total degree $m>1$ in n variables, and $H(f)$ its Hessian. Let ϕ by any analytic function. Prove that

$$H[\phi(f)] = H(f)\cdot\left[\left(\frac{\partial\phi}{\partial f}\right)^n + \frac{mf}{m-1}\,\frac{\partial^2\phi}{\partial f^2}\left(\frac{\partial\phi}{\partial f}\right)^{n-1}\right].$$

In particular when $\phi(f)$ is f^r, we have Mrs. Ballantine's generalization of Problem 2908 proposed by Professor Dickson [*1921*, 326; *1923*, 41].

Again, when $\phi(f)=\log f$, we have

$$H(f) = (1-m)f^nH[\log f],$$

which is a generalization of Exercise 22 of Sir Thomas Muir's "*Budget of Exercises on Determinants*" printed in this MONTHLY [*1922*, 10].

Solution by the Proposer.

For brevity of writing set $\phi'=d\phi/df$, $\phi''=d^2\phi/df^2$, $f_i=\partial f/\partial x_i$, and $f_{ij}=\partial^2 f/\partial x_i\partial x_j$. Then

$$H[\phi(f)] = \left|\frac{\partial^2\phi(f)}{\partial x_i\partial x_j}\right| = \left|\,\phi' f_{ij} + \phi'' f_i f_j\,\right|. \tag{1}$$

This can be expanded by columns into a sum of 2^n determinants. But, since all of these which involve more than one column of elements of the type $\phi'' f_i f_j$ are zero, we may write

$$H[\phi(f)] = (\phi')^n H(f) + (\phi')^{n-1}\phi''\sum f_i \begin{vmatrix} f_{11}\cdots f_1\cdots f_{1n} \\ \cdots\cdots\cdots \\ f_{n1}\cdots f_n\cdots f_{nn} \end{vmatrix}, \tag{2}$$

where the ith column contains the first partial derivatives of f, and the summation is for $i=1, 2, \cdots, n$. By Euler's theorem

$$f_j = \frac{1}{m-1}\sum_{k=1}^{n} x_k f_{jk},$$

and, by the use of this substitution in the ith column, the determinant in (2) expands into n determinants all of which but one is zero. The remaining one is $(m-1)^{-1}x_iH(f)$. Therefore

$$H[\phi(f)] = (\phi')^nH(f) + (\phi')^{n-1}\phi''(m-1)^{-1}H(f)\sum x_if_i.$$

Since $\sum x_if_i=mf$, we have the desired result

$$H[\phi(f)] = H(f)[(\phi')^n + m(m-1)^{-1}f\cdot(\phi')^{n-1}\phi''].$$

3276[1927, 381]. *Proposed by L. L. Silverman and J. Tamarkin.*

Prove that if ν is an integer greater than or equal to 1, then

$$\int_0^\infty \frac{(1+z)^{-\nu}dz}{\log^2 z + \pi^2} = (-1)^{\nu-1}\int_0^1 \binom{t}{\nu}dt, \text{ where } \binom{t}{\nu} = \frac{t(t-1)\cdots(t-\nu+1)}{\nu!}.$$

II. *Solution by G. Szegö, Königsberg, Germany.*

Consider the function

$$F(z) = \frac{(1+z)^{-\nu}}{\log z - i\pi} (0 < \Im \log z < 2\pi;\ \nu = 1, 2, \cdots)$$

and integrate it over the contour C consisting of two circles C_r, C_R about the origin of radii r and R respectively, and of a segment of the axis of reals between $z=r$, $z=R$, described twice in two opposite directions. It is immediately seen that the contribution of the circles C_r, $C_R \to 0$ as $r \to 0$, $R \to \infty$, while the contribution of the two-fold integration along the rectilinear part of the contour C tends to

$$\int_0^\infty \frac{(1+x)^{-\nu}}{\log x - i\pi}dx - \int_0^\infty \frac{(1+x)^{-\nu}}{\log x + i\pi}dx = 2i\pi\int_0^\infty \frac{(1+x)^{-\nu}}{\log^2 x + \pi^2}dx.$$

Since the integrand $F(z)$ is single-valued and analytic in C, except for a pole at $z=-1$, the integral of the problem will be equal to the residue of $F(z)$ at $z=-1$. To compute this residue we set $z+1=\zeta$, whence

$$F(z) = \frac{\zeta^{-\nu}}{\log(\zeta - 1) - i\pi} = \frac{\zeta^{-\nu}}{\log(1-\zeta)} = G(\zeta)$$

and compute the residue of $G(\zeta)$ at $\zeta = 0$. But this, in view of the expansion

$$\frac{\zeta}{\log(1-\zeta)} = \sum_{\nu=0}^{\infty} a_\nu \zeta^\nu,$$

will coincide with the coefficient a_ν.

A solution of this problem by J. Tamarkin appeared in 1928, page 497.

3433. [1930, 315]. *Proposed by Thurman Andrew, Jamaica Plain, Mass.*

Consider three elastic rods of equal lengths and cross sectional areas. The cross sections have the forms of a circle, an equilateral triangle and a rectangle whose long side is twice the diameter of the circle. What will be the ratio of the torques required to turn them through a given angle about their axes? What is the shape of the solid rod which requires the greatest torque? The elastic limit is not to be exceeded.

Solution by D. L. Holl, Ames, Iowa

The solution of the torsion problem[1] subject to the condition of only a pure

[1] A. E. H. Love, *The Mathematical Theory of Elasticity*, 4th edition (1927); p. 310; *Handbuch der Physik*, Bd. VI (1928), p. 143; J. Prescott, *Applied Elasticity* (1924), p. 188.

couple over the cross section of the rod and no action on the sides, requires the solution of $\nabla^2\xi = -2\tau$ such that $\xi = 0$ on the contour. The torquing couple is then $Q = 2\mu\int\xi dA$ where the integration is carried out over the cross section of the prism, and the symbols are defined as follows: τ is twist per unit length of rod, and μ is the modulus of shear or $\mu = E/2(1+\sigma)$ with E the Young's modulus, and σ is Poisson's ratio.

For the circular rod of radius a, the solution is $\xi = \frac{1}{2}\tau(a^2 - r^2)$ and $Q_c = \frac{1}{2}\tau\pi\mu a^4$.

For the equilateral triangle,

$$\xi = \tfrac{1}{6}\tau a_1^{-1}(3xy^2 - x^3 + 4a_1^3) - \tfrac{1}{2}\tau(x^2 + y^2)$$

and

$$Q_T = (9\sqrt{3}/5)\mu\tau a_1^4,$$

where a_1 is the radius of the inscribed circle. To satisfy the condition of equal cross sections, $3\sqrt{3}a_1^2 = \pi a^2$; hence $Q_T = (\pi^2\sqrt{3}/15)\mu\tau a^4$.

For a rectangle of sides $2a$, $2b$, the solution is

$$\xi = \tau(a^2 - x^2) - \frac{32}{\pi^3}\tau a^2 \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cosh[(2n-1)\pi y/2a]\cos[(2n-1)\pi x/2a]}{(2n-1)^3\cosh[(2n-1)\pi b/2a]},$$

and

$$Q_R = \frac{16}{3}a^3 b\mu\tau - \left(\frac{4}{\pi}\right)^5 a^4\mu\tau\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\tanh[(2n-1)\pi b/2a].$$

This series is rapidly convergent and, when b_1 and a_1 denote the longest and shortest dimensions, for $b_1 > 3a_1$ the following expression is sufficiently accurate:

$$Q_R = \tfrac{1}{3}a_1^3 b_1\mu\tau\left(1 - 0.630\frac{a_1}{b_1}\right).$$

To satisfy equal cross sections, $b_1 = 4a$, $a_1 = \frac{1}{4}\pi a$ in terms of radius of circle. Then

$$Q_R = \frac{\pi^3}{48}\mu\tau a^4\left(1 - 0.630\frac{\pi}{16}\right).$$

Hence

$$Q_C : Q_T : Q_R = 1 : \frac{2\sqrt{3}}{15}\pi : \frac{\pi^2}{24}\left(1 - 0.630\frac{\pi}{16}\right)$$

$$= 1 : 0.725 : 0.360.$$

In each of the above cases, the origin of coördinates was taken at the centroid of the section.

To find the section which requires the greatest torque for a given twist τ per unit length, is to solve a problem of calculus of variations; the section can be shown to be a circle.

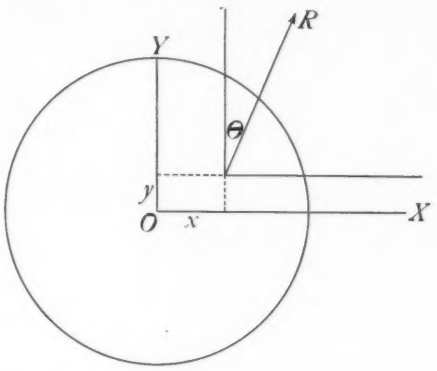
Intuitively it might be argued that since the rods have equal volume, and since the maximum stresses occur on the boundary at points nearest the center of twist, the circle, by reason of its compact area and uniformly increasing stresses towards the periphery, will absorb the most elastic energy per unit length, hence requiring the greatest torque per unit of twist τ.

3462. [1930, 508]. *Proposed by R. C. Colwell and O. R. Ford, West Virginia University.*

A circular hoop, which is free to move on a smooth horizontal plane, has sliding on it a small ring of $1/n$th its mass, the coefficient of friction between the two being μ. Initially the hoop is at rest and the ring has an angular velocity ω round the hoop. Show that the ring comes to rest relative to the hoop after a time $(1+n)/\mu\omega$.

Solution by the Proposer

Take the axes of x and y at the original position of the center of the hoop so that the coördinates x, y represent the position of the center at any time t.



The equations of motion are

$$m\ddot{x} = R\sin\theta + \mu R\cos\theta, \tag{1}$$

$$m\ddot{y} = R\cos\theta - \mu R\sin\theta, \tag{2}$$

$$\frac{m}{n}\frac{d^2}{dt^2}[x + a\sin\theta] = -R\sin\theta - \mu R\cos\theta, \tag{3}$$

$$\frac{m}{n}\frac{d^2}{dt^2}[y + a\cos\theta] = -R\cos\theta + \mu R\sin\theta. \tag{4}$$

From (1) and (3), with one integration,

$$\frac{m(n+1)}{n}\dot{x} + \frac{m}{n}(a\cos\theta\cdot\dot{\theta}) = C$$

when

$$t = 0,\ \dot{x} = 0,\ \cos\theta = 1,\ \dot{\theta} = \omega;\ \text{therefore } C = ma\omega/n$$

or

$$(n + 1)\dot{x} + a\cos\theta\cdot\dot{\theta} = a\omega.$$

From this equation

$$(n + 1)x + a\sin\theta = a\omega t. \tag{5}$$

From (2) and (4),

$$\frac{m}{n}\frac{d^2}{dt^2}(y + a\cos\theta) = -m\ddot{y},$$

or

$$\frac{m}{n}\ddot{y} - \frac{m}{n}\frac{d}{dt}(a\sin\theta\cdot\dot{\theta}) = -m\ddot{y},$$

or

$$\frac{m(n+1)}{n}\dot{y} - \frac{m}{n}a\sin\theta\cdot\dot{\theta} = C'.$$

When

$$t = 0,\ \dot{y} = 0,\ \sin\theta = 0;\ C' = 0;$$

hence

$$(n + 1)\dot{y} - a\sin\theta\cdot\dot{\theta} = 0.$$

Integrating,

$$(n + 1)y = a(1 - \cos\theta). \tag{6}$$

The position of the center of gravity of the system is determined by the equations

$$\bar{x} = x + [a/(n + 1)]\sin\theta,\quad \bar{y} = y + [a/(n + 1)]\cos\theta.$$

Using these equations in (5) and (6), we have

$$\bar{x} = [a/(n + 1)]\omega t,\quad \bar{y} = a/(n + 1).$$

This equation shows that the center of gravity of the whole system travels along a straight line in the x direction with uniform velocity $a\omega/(n+1)$ and at a distance $a/(n+1)$ from that axis.

We may now write out the equations corresponding to (1), (2), (3), and (4) with regard to the center of gravity. Then

$$m\ddot{x} = m\frac{d^2}{dt^2}\left[\bar{x} - \frac{a}{n+1}\sin\theta\right] = R\sin\theta + \mu R\cos\theta, \tag{7}$$

$$m\ddot{y} = m\frac{d^2}{dt^2}\left[\bar{y} - \frac{a}{n+1}\cos\theta\right] = R\cos\theta - \mu R\sin\theta, \tag{8}$$

$$\frac{m}{n}\frac{d^2}{dt^2}\left[\bar{x}+\frac{na\sin\theta}{n+1}\right]=-R\sin\theta-\mu R\cos\theta, \tag{9}$$

$$\frac{m}{n}\frac{d^2}{dt^2}\left[\bar{y}+\frac{na\cos\theta}{n+1}\right]=-R\cos\theta+\mu R\sin\theta. \tag{10}$$

From (7) and (9),

$$\frac{m}{n}a\frac{d^2}{dt^2}\sin\theta=-R\frac{(n+1)}{n}(\sin\theta+\mu\cos\theta). \tag{11}$$

From (8) and (10),

$$\frac{m}{n}a\frac{d^2}{dt^2}\cos\theta=-R\frac{(n+1)}{n}[\cos\theta-\mu\sin\theta]. \tag{12}$$

Dividing (11) by (12), we obtain

$$\frac{\cos\theta\cdot\ddot{\theta}-\sin\theta\cdot\dot{\theta}^2}{-\sin\theta\cdot\ddot{\theta}-\cos\theta\cdot\dot{\theta}^2}=\frac{\sin\theta+\mu\cos\theta}{\cos\theta-\mu\sin\theta}.$$

This reduces to the simple form

$$\ddot{\theta}+\mu\dot{\theta}^2=0.$$

The integration of this equation gives

$$\mu t=\frac{1}{\dot{\theta}}-\frac{1}{\omega}. \tag{13}$$

The law of conservation of angular momentum about the center of gravity gives $I_1\,\omega=I_2\dot{\theta}$, where $I_1=ma^2/(n+1)$ is the initial moment of inertia and $I_2=m(n+2)a^2/(n+1)$ is the final moment of inertia when the hoop is set into rotation. From these equations

$$\dot{\theta}=\omega/(n+2). \tag{14}$$

Using (14) in (13) we have

$$t=(n+1)/\mu\omega.$$

Also solved by T. W. Edmondson and T. L. Smith.

3469[1931, 50]. *Proposed by V. M. Spunar, Chicago, Illinois.*

A constant length, PQ, is measured along the tangent at any point, P, on a curve; give a geometrical construction for the center of curvature of the locus of the point Q. Also if PQ' be measured equal to PQ in the opposite direction along the tangent, prove that the point P and the centers of curvature of the loci of Q and Q' lie in a straight line.

Solution by R. Goormaghtigh, Bruges, Belgium.

Let (Q) be the locus of Q and q the center of curvature of (Q) at Q. The length PQ being constant, the normal to (Q) at Q passes through the center of curvature C of the given curve (P) at P. If further the normal at C to the evolute (C) of (P) cuts at T the perpendicular to Qq at q, and if C' is the center of curvature of (C) at C, we have

$$d(P)/d(Q) = PC/QC, \quad d(Q)/d(C) = Qq/CT, \quad d(C)/d(P) = CC'/PC,$$

and therefore

$$QC : Qq = CC' : CT.$$

Hence the following construction:

If the perpendicular at C to QC cuts QC' at S, the straight line joining S to the mid-point of CC' passes through the center of curvature of (Q).

Further, the tangents at Q, Q' and C to the loci (Q), (Q') and (C) of these points are concurrent. Hence, if Q_1, Q_1' and C_1 are corresponding points on (Q), (Q') and (C) having as limits Q, Q' and C, the triangles $QQ'C$ and $Q_1Q_1'C_1$ are homological and therefore the limits of the intersections of their corresponding sides, i.e. the centers of curvature of (Q), (Q') and the point P, are collinear.

A Note by Otto Dunkel: Since this interesting solution is rather condensed, we add reasons for its statements, which, while not being the simplest, result in a uniform manner from two known relations. If a straight line AB of length l varies so that A moves on the curve (A) and B on (B), then

$$ds \cos A + dl = d\sigma \cos B, \tag{1}$$

$$ds \sin A + l\,d\theta = d\sigma \sin B, \tag{2}$$

where θ is the angle that AB makes with a fixed straight line; A is the angle that the element of arc ds of (A) makes with the direction AB, and B is the angle that $d\sigma$ of (B) makes with the same direction.

If $l = PQ = c$, constant; angle $A = 0$; $A \equiv P$; $B \equiv Q$; we have

$$ds = d\sigma \cos Q, \tag{3}$$

$$c\,d\theta = d\sigma \sin Q. \tag{4}$$

If the normals at Q and P meet in C, (3) and (4) give $ds/d\theta = c \cot Q = CP$. Hence C is the center of curvature of (P) at P. The equation (3) may be written $d(P)/d(Q) = PC/QC$.

For the curves (q) and (Q) we have $qQd\theta = d(Q)$. (θ is not the same angle as above). For (q) and (C) we have from (2) $qCd\theta = d(C) \sin C = d(C)qC/TC$, or $TCd\theta = d(C)$. These two results give $Qq/CT = d(Q)/d(C)$.

Since C' is the center of curvature of (C), $C'Cd\theta = d(C)$; also $CPd\theta = d(P)$. Hence $CC'/PC = d(C)/d(P)$.

A proof of the construction is as follows: Let qC' and qT cut QP in M and N. Then $CC'/CT = QM/QN$, and hence $QC/Qq = QM/QN$, since it was proved that the ratios on the left are equal. Hence qT is parallel to CM, and S must lie on CM. The complete quadrilateral determined by C, C', M, Q has q and ∞ for its other two vertices. Hence qS bisects CC' and QM.

The argument for the last part may be stated in this manner. The tangents, at Q and Q' meet in O on CP. If P_1 on (P) is very near P, the corresponding points Q_1, Q; Q_1', Q'; C_1, C are very near each other. Except for infinitesimals of higher order than the first, Q_1 may be regarded as lying upon OQ; Q_1' on OQ'; C_1, on OC. The nearer P_1 is to P, the more nearly true are these statements. Hence at the limit the three points $(QC,\ Q_1C_1)$; $(Q'C,\ Q_1'C_1)$; $(QQ',\ Q_1Q_1')$ lie in a straight line, i.e., q, q', P lie in a straight line.

A Second Proof. The geometric reasoning may be put in a form somewhat different from the above. Since the normals at P and Q meet in a point C, this point is the instantaneous center of rotation of $PC=c$. From the elementary geometry proof of this fact, it will follow that $PC=r$ is the radius of curvature at P, and that $ds/r=d\sigma/k$, where $k=QC$. Let $\angle PQC=\phi$, then from $c\tan\phi=r$, we find $c\sec^2\phi d\phi=dr$, or $k^2d\phi=cdr$. Since dr is the element of arc of the evolute (C), we have $\bar{r}d\theta=dr$, where $\bar{r}=CC'$; also $rd\theta=ds$. Combining these results we have

$$ds/d\sigma = r/k, \quad d\phi/ds = c\bar{r}/(rk^2). \tag{1}$$

Consider the two neighboring figures PQC and $P_1Q_1C_1$, and let $\Delta\psi$ denote the angle between QC and Q_1C_1. An inspection of the figure shows that $\Delta\psi=\Delta\theta-\Delta\phi$. If we set $Qq=R$, then

$$\frac{1}{R}=\frac{d\psi}{d\sigma}=\frac{ds}{d\sigma}\left(\frac{d\theta}{ds}-\frac{d\phi}{ds}\right),$$

and this reduces by (1) to

$$\frac{k}{R}=1-\frac{c\bar{r}}{k^2}, \text{ or } \frac{R-k}{R}=\frac{c\bar{r}}{k^2}=\frac{Cq}{Qq}. \tag{2}$$

To locate q draw CM perpendicular to QC cutting QP at M. From the figure we see that $c/k=k/QM$, and this with the last result in (2) gives

$$Cq/Qq = CC'/QM. \tag{3}$$

This shows how to construct q. Draw MC' cutting QC in a point which (3) shows must be q.

In the isosceles triangle QCQ', q lies on QC and q', on $Q'C$. Take $\bar{q}$ on QC so that $Q\bar{q}=Q'q'$. Then $P(Q, \bar{q}, C, q)$ is a harmonic pencil, since for q' we use $-c$ in place of c in (2). Since PC is perpendicular to PQ, the ray PC bisects the angle $\bar{q}Pq$. This shows that P, q, q' lie in a straight line.

The results in (2) may also be derived rather simply by vector analysis.

3476 [1931, 111]. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Given the cubic, $x^3+ax+b=0$, with rational roots α, β, γ; show that the equation $\alpha y^2+\beta y+\gamma=0$ also has rational roots.

Solution by Marjorie E. Halliwill, A Student at Akron University, Akron, Ohio.

Since $\alpha+\beta+\gamma=0$, the quadratic may be written $(y-1)(\alpha y-\gamma)=0$. One root is obviously rational, and the second is also rational from the given conditions.

Also solved by R. P. Agnew, H. W. Bailey, A. W. Bear, W. R. Church, O. C. Collins, A. G. Clark, Ralph Deutsch, Edward Fleisher, D. F. Gunder, J. D. Hill, J. D. Leith, Gertrude I. Mc Cain, G. I. Miller, R. E. Moritz, J. H. Neelley, A. Pelletier, E. A. Rasor, P. L. Rea, J. Rosenbaum, Wallace Smith, Mrs. B. J. Smyth, F. Underwood, B. F. Yanney, G. L. Weaver, Kamcheung Woo, and the Proposer.

3477[1931, 111]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

If two points, harmonically separated by the centers of two unequal circles (spheres), are diametrically opposite on a circle (sphere) coaxial with the given circles (spheres), these two points are the centers of similitude of the two given circles (spheres).

Note: This is the converse of the property of the circle (sphere) of similitude. (See, for instance, Nathan Altshiller-Court, *College Geometry*, p. 196.)

Solution by Edwin J. Purcell, University of Colorado.

Given two circles S and S' with radii r and r' respectively, $r \neq r'$. Without loss of generality we may take their centers O and O' as the origin and the point $(a', 0)$ respectively in rectangular Cartesian coördinates. Their equations are

$$S \equiv x^2 + y^2 - r^2 = 0, \text{ and } S' \equiv (x - a')^2 + y^2 - r'^2 = 0.$$

A third circle S'', coaxial with S and S', will have its center collinear with O and O'. Its equation is

$$(1) \quad S'' \equiv S + kS' \equiv (x^2 + y^2 - r^2) + k(x^2 - 2a'x + a'^2 + y^2 - r'^2) = 0,$$

where k is an arbitrary constant. The points of intersection, B and A, of S'' with the line of centers, $y = 0$, are given by the values of x in

$$(2) \qquad x = \frac{a'k \pm \sqrt{(-ka'^2 + kr'^2 + r^2 + k^2r'^2 + kr^2)}}{(1 + k)}.$$

The problem imposes the condition that the four points O, B, O', A, be harmonic. Upon substitution and simplification, this condition gives

$$(3) \qquad (1 + k)(kr'^2 + r^2) = 0.$$

It is to be shown that B and A are the centers of similitude of the circles S and S'. Now if k equals -1, the equation (1) gives a straight line, which is to be excluded. Therefore, from (3),

$$(4) \qquad k = -r^2/r'^2.$$

Substituting (4) in (2) and simplifying, the coördinates of B and A are

$$x = \frac{a'r}{r + r'}, \; y = 0; \text{ and } x = \frac{a'r}{r - r'}, \; y = 0;$$

and these are precisely the coördinates of the centers of similitude of the two given circles, S and S' (see, for instance, G. Salmon: *Conic Sections*, 1904, page 105).

The proof of the corresponding theorm for three dimensions will be similar to that given above if we take the two given spheres as having their respective centers at (0, 0, 0) and (a', 0, 0) and consider the section made by the plane $z=0$.

Editorial Note: The proof above may be put in a different form by replacing (1) by the known equivalent fact that the lengths of the tangents from B to the given circles have the same ratio as the corresponding tangents from A. Hence

$$\frac{\overline{OB}^2 - r^2}{\overline{O'B}^2 - r'^2} = \frac{\overline{OA}^2 - r^2}{\overline{O'A}^2 - r'^2}.$$

From the harmonic property we have also $OB/OA = -O'B/O'A = p$. These equations combined give $OA/O'A = -OB/O'B = r/r'$, and the desired result follows at once.

Also solved by A. Pelletier and F. Underwood.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The preliminary official program of the International Mathematical Congress to be held in Zurich in 1932, which has just been issued, indicates that the sessions of the Congress will be on September 4–12, that all countries will be represented, and that titles of papers intended for presentation at the Congress should be sent to the Secretary of the International Mathematical Congress, École Polytechnique Fédérale, Zurich. The president of the committee on organization of the Congress is Professor R. Fueter. It is the present intention also to hold the sessions of the International Mathematical Union, which is a distinct organization, and of which Professor W. H. Young is the president, during the time of the Congress.

In commemoration of the one hundredth anniversary of the death of Laplace (in 1827) a memorial momument will be erected in the town of Beaumont, his birthplace. Dedication will take place in the summer of 1932.

Professor R. L. Moore, of the University of Texas, who was appointed by the Council of the American Mathematical Society to the Travelling Lecturership for the current academic year, gave the first series of lectures on the Pacific Coast, on the following dates: University of California at Los Angeles, November 10, 12, 13; Stanford University, November 16, 17, 18; University of Washington, November 20; University of California, Berkeley, November 23,

24, 25. The titles of his lectures were: 1. *Foundations of point set theory;* 2. *Continuous curves;* 3. *Upper semi-continuous collections.* Professor Moore also gave an address entitled *A set of axioms for plane analysis situs,* by invitation of the program committee at the meeting of the Society held at the California Institute of Technology on November 28.

Professor Willem de Sitter, director of the Leyden Observatory, has been elected an honorary member of the American Astronomical Society.

Professor R. A. Millikan, of the California Institute of Technology, has been made a Knight of the Legion of Honor by the French Government.

The following have been awarded National Research Fellowships in mathematics for the academic year 1931–32; R. P. Agnew, E. F. Beckenbach, Leonard Carlitz, Benedict Cassen, Max Coral, H. B. Curry, J. L. Dorroh, A. L. Foster, D. H. Lehmer, S. B. Littauer, E. J. McShane, G. W. Mendel, C. B. Morrey, A. L. O'Toole, A. E. Ross, Wladimir Seidel, W. J. Trijitzinsky, Hassler Whitney, S. S. Wilks, Jacob Yerushalmy, Leo Zippin. This list includes renewals.

Dr. W. S. Adams, director of the Mount Wilson Observatory, has been elected president of the American Astronomical Society.

Dr. R. P. Agnew has been appointed assistant professor of mathematics at Cornell University.

Dr. W. D. Baten has been promoted to an assistant professorship of mathematics at the University of Michigan.

Mr. William Beverly has been promoted to an assistant professorship of mathematics at Lafayette College.

Mr. W. M. Birchby has been promoted to an assistant professorship of mathematics at the California Institute of Technology.

Mr. J. C. Brixey, of the University of Oklahoma, has been promoted to an assistant professorship of mathematics.

Associate Professor Thomas Buck, of the University of California, Berkeley, has been promoted to a professorship of mathematics.

Mr. Earl Church, of Syracuse University, has been promoted from an assistant professorship of mathematics to an associate professorship of photogrammetry.

Professor Elbert H. Clarke, head of the department of mathematics and astronomy in Hiram College, Hiram, Ohio, has been granted a six months leave of absence for study at the University of Göttingen.

Dr. C. M. Cleveland has been appointed adjunct professor of applied mathematics at the University of Texas.

Assistant Professor Tobias Dantzig, of the University of Maryland, has been promoted to an associate professorship of mathematics.

Dr. M. S. Demos, of Columbia University, has been appointed assistant professor of mathematics at the University of New Hampshire.

Dr. W. L. Duren, of the College of the City of Detroit, has been appointed assistant professor of mathematics at Tulane University.

Dr. N. C. Fisk, of the University of Michigan, has been appointed dean of the Itasca Junior College, Coleraine, Minnesota.

Associate Professor M. C. Foster has been promoted to a professorship of mathematics at Wesleyan University.

Miss Marjorie L. Heckel has been appointed instructor at the University of South Dakota.

Dr. P. E. Hemke has been appointed associate professor of mechanics at the Case School of Applied Science.

Mr. A. S. Householder has been appointed to an assistant professorship of mathematics at Washburn College.

Dr. Gertrude I. McCain has been appointed professor of mathematics at Marymount College, Salina, Kansas.

Dr. G. F. McEwen has been appointed chairman of the newly created committee of the American Geophysical Union on evaporation from free water surfaces.

Miss Martha P. McGavock has been promoted to an associate professorship of mathematics at Rockford College.

Dr. L. T. Moore has been appointed assistant professor of mathematics at Brooklyn College of the City of New York.

Associate Professor D. S. Morse has been promoted to a professorship of mathematics at Union College.

Dr. C. V. Newsom has been appointed associate professor of mathematics at the University of New Mexico.

Associate Professor C. S. Porter, of the department of mathematics at Amherst College, has been appointed acting dean of the College.

Miss Mary J. Quigley, of Boston Teachers College, has been promoted to a professorship of mathematics.

Mr. H. P. Rogers, of the University of Illinois, has been appointed assistant professor of mathematics at Kent State College, Kent, Ohio.

Dr. J. A. Shohat has been promoted to an assistant professorship of mathematics at the University of Pennsylvania.

Dr. L. B. Slichter, of the California Institute of Technology, has been appointed associate professor of mathematics at the Massachusetts Institute of Technology.

Associate Professor R. G. Smith, of Kansas State Teachers College, Pittsburg, has been promoted to a professorship.

Assistant Professor Pauline Sperry, of the University of California, Berkeley, has been promoted to an associate professorship of mathematics.

Dr. Elizabeth Stafford Sokolnikoff has been apointed instructor at the University of Wisconsin.

Dr. F. G. Williams has been appointed professor of mathematics at Susquehanna University.

Dr. R. C. Yates has been appointed assistant professor of mathematics at the University of Maryland.

Richard Bolling Dunn, instructor in mathematics at Yale University, has died recently.

Dr. Stephen Marshall Hadley, who was for many years dean and professor of mathematics at Penn College, Oskaloosa, Iowa, died at Whittier, California, on November 10, 1931. He was a charter member of the Mathematical Association.

Dr. Charles Haseman, professor of mathematics and mechanics at the University of Nevada, died on July 9, 1931, after a year's illness. He was a charter member of the Association.

Mr. Alexander Knisely of Columbia City, Indiana, a charter member of the Association, died September 29, 1931.

Dr. Bessie I. Miller, of the University of Illinois, died February 4, 1931. Dr. Miller was a charter member of the Association.

Professor C. L. E. Moore, of the Massachusetts Institute of Technology, died on December 5, 1931. He was a charter member of the Association.

Professor C. W. Watts, of Virginia Military Institute, a charter member of the Association, died on July 10, 1931.

The Chauvenet Prize

In the year 1925, the MATHEMATICAL ASSOCIATION OF AMERICA established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of.....Dollars, to be known as the.....Fund, and to be used for[2] { Endowment—the income only of which may be expended. / Special Projects—for which both principal and income may be expended. }

Witness:..... Signature.....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Sixteenth Summer Meeting of the Association, Los Angeles, Calif., Aug. 29-30, 1932.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1932 and reported to the secretary.

ILLINOIS, Urbana.
INDIANA.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, College Park, Md., May.
MICHIGAN.
MINNESOTA.
MISSOURI.
NEBRASKA.
OHIO, Columbus, Ohio, April 7.
PHILADELPHIA, Philadelphia, Pa., Nov. 26.
ROCKY MOUNTAIN, Laramie, Wyo.
SOUTHEASTERN.
SOUTHERN CALIFORNIA, San Diego, March 26.
TEXAS, Austin, Jan. 30.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE SIXTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The sixth annual meeting of the Philadelphia Section of the Mathematical Association of America was held in the Packard Laboratory of Lehigh University on Saturday, November 28, 1931. Professor Fort of Lehigh University presided at both sessions.

The attendance was thirty-three, including the following twenty-four members of the Association: S. S. Cairns, P. A. Caris, J. W. Clawson, Arnold Dresden, Tomlinson Fort, J. R. Kline, M. S. Knebelman, P. A. Knedler, P. V. Kunkel, V. V. Latshaw, F. L. Manning, A. E. Meder, Jr., G. E. Raynor, J. B. Reynolds, J. A. Roulton, C. A. Rupp, J. A. Shohat, C. A. Shook, L. L. Smail, W. M. Smith, Alexander Tartler, Willis Whited, Clement Weinstein, A. H. Wilson.

At the business meeting the following officers were chosen for next year: Chairman, Arnold Dresden, Swarthmore College; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Professors Dresden, Caris, Smith, and Morris. The next meeting of the Section will be held on Saturday, November 26, 1932, at Philadelphia.

The following papers were presented:

1. "Redundant co-ordinates" by Professor C. A. Rupp, Pennsylvania State College.
2. "On some fundamental conceptions in the theory of infinite processes" by Professor L. L. Smail, Lehigh University.
3. "Italy and geometry" by Professor W. M. Smith, Lafayette College.
4. "Different kinds of curvature" by Professor M. S. Knebelman, Princeton University.
5. "Swarthmore honors course in mathematics" by Professor Arnold Dresden, Swarthmore College.

Abstracts of the papers follow:

1. The choice, for the co-ordinates of a geometric element, of a set of quantities whose number exceeds, by more than one, the number of degrees of freedom of the element, may have the advantage of yielding superior ease in the algebraic manipulations of the co-ordinates. The price paid for the gain in manoeuverability is that the task of finding multipliers in questions of linear dependence is complicated by the fact that the superfluous co-ordinates are functions of the essential co-ordinates. In spite of such complications, it is possible, in some cases to make the machinery of linear dependence work. This paper draws some geometric consequences of the linear dependence of flat spaces in a space of n dimensions.

2. In this paper Professor Smail discussed the definitions of infinite series and of convergence, and the fundamental significance of the concept of summability of infinite processes. He criticized the usual definitions of the concept